

# Fractional Charge

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## Abstract

The origin and quantum status of Fractional Charge in polyacetylene and field theory are reviewed, along with reminiscences of collaboration with John Bell on the subject.

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## I. INTRODUCTION

I cannot claim the privilege of having known Professor John Bell as intimately or for as long as some of the other participants in this conference to honour his memory. But there was a period of about 3 years from 1982 to 1985 when we interacted a fair amount and conducted (mostly at long distance) a collaboration trying to understand the then recently discovered phenomenon of fractional charge. During this period we had the opportunity of hosting him at my Institute in Bangalore. He came home for dinner and met my family. I in turn went to CERN , first to write up our work and later for a long visit, when I had the pleasure of meeting his wife Dr. Mary Bell and was the beneficiary of their kindness in many ways.

During this interaction I developed , like so many people before me, great respect for Professor Bell, not just for his great originality and prowess as a theoretical physicist, but equally for the precision of thought and language that he demanded of himself and of those who were fortunate enough to work with him. John Bell was more than an intellectual force in physics— his was a moral presence. One can ill-afford the loss of such a moral presence in today's world, and it is a great tragedy that he was taken away from us when we continue to need him so much.

Long before I met John, I had of course heard of him and his many important contributions. In fact the very first time I had to study his work was in the early 'sixties, when I was a foot-soldier in Professor Hans Bethe's army attacking the nuclear matter problem. All of you know that John had worked on a wide range of subjects, but not all might know that one of his early interests was nuclear matter, on which he had co-authored a long review article with E.J.Squires way back in 1961. A few years after that as I gravitated towards particle physics and quantum field theory I had to study in great detail his landmark discovery with Roman Jackiw of the axial current anomaly.

Despite such overlap of interests, I met him only much later in life when he spent a couple of weeks at the Indian Institute of Science in Bangalore where I was then working. A

colleague introduced us and he made the usual polite enquiry – "What are you working on these days ?" I thought that rather than inflict on him the different problems I was working on at that time, I would instead get his "take" on the very interesting but puzzling claims of fractionally charged states that had recently emerged.

As it turned out, he had not heard about these developments. "You mean fractionally charged like quarks ? Isn't that by now old and well established ? ", says John , a slight note of wariness creeping into his voice. " No, no ", I hasten to add, "I don't mean quarks . By fractionally charged I refer to states with fractional eigenvalues of the Number Operator. These have been discovered not just theoretically in soliton states of field theory models, but are claimed to be present in down to earth systems like polymer chains . What I don't understand is how a real polymer made of some finite number of electrons could, no matter how it twists itself into a soliton configuration, carry a fractional number of electrons." This produced a flicker of a smile from him and I knew I had caught his interest. "Would you like to tell me about it ?", he asked. So I ushered him into my office and closed the door . This was a lovely opportunity , having trapped the distinguished John Bell in one's office with all avenues of escape sealed off, to summarize for him *ab initio* the curious phenomenon of fractional charge and get his opinion.

Let me begin the next section with a pedagogical expansion of what I described to John in the fading light of that winter afternoon in Bangalore.

## II. FRACTIONAL CHARGE IN FIELD THEORY

The phenomenon of fractional charge was first discovered by Jackiw and Rebbi in their pioneering work in 1976 in the context of soliton states in quantum field theory [1]. Three years after the Jackiw-Rebbi work, but quite independently and in an entirely different context, Su, Schrieffer and Heeger [2], argued that a similar phenomenon can occur for soliton states in the long chain molecule *trans*-Polyacetylene. See also the work by Rice [3]. Subsequently Jackiw and Schrieffer [4] wrote a paper drawing attention to the similarity

between these results discovered in entirely different areas of physics.

Let us begin with the Jackiw-Rebbi work . They discussed the phenomenon in both one- dimensional and three-dimensional models. Let me for simplicity discuss just the one-dimensional example, which contains all the essential ideas. Consider in (1+1) dimensions a Fermi field  $\Psi(x, t)$  coupled to a scalar field  $\Phi(x, t)$  through a Lagrangian density

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F \tag{1}$$

where,

$$\mathcal{L}_B = \frac{1}{2g^2} \left[ \left( \frac{\partial \Phi}{\partial t} \right)^2 - \left( \frac{\partial \Phi}{\partial x} \right)^2 - (1/2)(\Phi^2 - 1)^2 \right] \tag{2}$$

and

$$\mathcal{L}_F = \bar{\Psi} \left( i \partial_\mu \gamma^\mu - m \Phi(x, t) \right) \Psi \tag{3}$$

This example corresponds to a quartic double well potential in  $\Phi$ . It is straightforward to generalise the results to other potentials  $U(\Phi)$  with symmetric double wells. The phenomenon of fractional charge occurs in the soliton sector of this system. To a wider readership it may be helpful to recall what is meant by the soliton sector and how fermi fields are treated in such sectors ( [5]).

The bosonic sub-system  $\mathcal{L}_B$ , in the absence of the Fermi field, has as its field equation

$$\left( \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \phi - \phi + \phi^3 = 0 \tag{4}$$

This has the following two lowest energy classical solutions :

$$\phi(x, t) = \pm 1 \tag{5}$$

As is well known the presence of these two degenerate classical solutions indicates spontaneous breaking of the  $\Phi \leftrightarrow -\Phi$  symmetry of the Lagrangian. Around each of these classical solutions a separate vacuum state and a whole tower of Fock states can be built corresponding to the two phases this system permits. We will call these the vacuum sectors.

In addition this system also has two other static (time independent) solutions. Those are the topological soliton solutions – the so called kink solution and its reflected partner, the anti-kink, given respectively by

$$\phi_S(x) = \pm \tanh(x/\sqrt{2}) \quad (6)$$

As per the general theory of semi classical quantization of quantum fields, one can build two other separate towers of states, one around each of these soliton solutions. The topological index  $n = \phi(\infty) - \phi(-\infty)$  becomes a superselection quantum number upon quantization and prevents any states from these soliton sectors from decaying into the vacuum sector. In short, we have four sectors of states for this bosonic system : two are the vacuum sectors built around  $\phi = \pm 1$  and the other two are the soliton and antisoliton sectors built around the kink and antikink solutions in eq (6). For more details on soliton sectors of states and their properties see ([5]).

Now consider the full system in (1) including the Fermi field.

### A. Vacuum sector

To leading order in  $\hbar$  one can replace in each sector the Bose field operator  $\Phi$  occurring in the fermi field lagrangian (3) by the corresponding classical solution . Thus in the vacuum sector built around  $\phi = 1$ , the fermi lagrangian (3) reduces in leading order to

$$\mathcal{L}_F = \bar{\Psi} \left( i\partial_\mu \gamma^\mu - m \right) \Psi \quad (7)$$

This is just the free Dirac system with mass  $m$  discussed in textbooks. In order to contrast with what later happens in the soliton sector , let us recall why the Number (Total charge) operator has only integral eigenvalues for the free Dirac system. Let us denote by  $u_k(x)$  and  $\tilde{u}_k(x)$  the positive and negative energy spinorial solutions of the Dirac equation

$$\begin{aligned} (-i\alpha\partial_x + \beta m)u_k(x) &= E_k u_k(x) \\ \left( -i\alpha\partial_x + \beta m \right)\tilde{u}_k(x) &= -E_k \tilde{u}_k(x) \end{aligned} \quad (8)$$

where  $E_k = +\sqrt{k^2 + m^2}$  and spinor indices have been suppressed. The Dirac matrices for this 1+1 dimensional system can be taken to be  $\alpha = \sigma_2$  and  $\beta = \sigma_1$ . The Dirac field is expanded in terms of these solutions and the destruction operators  $b_k$  and  $d_k$  obeying the usual anticommutation rules .

$$\Psi(x, t) = \sum_k [b_k u_k e^{-iE_k t} + d_k^\dagger \tilde{u}_k e^{iE_k t}] \quad (9)$$

The vacuum state in the  $\phi = 1$  sector is given by the conditions

$$b_k |vac\rangle = d_k |vac\rangle = 0 \quad (10)$$

with all the bosonic oscillators being in the ground state.

Note that the third Pauli matrix  $\sigma_3$  acts as the charge conjugation matrix. It anti commutes with the Dirac hamiltonian in (8) and generates for every positive energy solution  $u_k(x)$  of energy  $E_k$  the corresponding negative energy solution of energy  $-E_k$  :

$$\sigma_3 u_k(x) = \tilde{u}_k(x) \quad (11)$$

Hence all modes of the expansion ( 9) come in pairs with positive and negative energy. There are no zero energy solutions in the free massive Dirac equation.

Finally consider the charge density operator (which is really the number density operator)

$$\rho(x, t) = \frac{1}{2} [\Psi^\dagger(x, t) , \Psi(x, t)] \quad (12)$$

This commutator form of  $\rho$  is standard; it is designed to be regularised and odd under charge conjugation . Inserting the mode expansion (9) and using the orthonormality of the Dirac solutions the total charge becomes

$$\begin{aligned} Q &\equiv \int dx \rho(x, t) \\ &= \frac{1}{2} \sum_k \left( [b_k^\dagger, b_k] + [d_k, d_k^\dagger] \right) \\ &= \sum_k \left( (b_k^\dagger b_k - 1/2) - (d_k^\dagger d_k - 1/2) \right) \\ &= \sum_k \left( b_k^\dagger b_k - d_k^\dagger d_k \right) \end{aligned} \quad (13)$$

Notice that the half-integers cancel term by term because of the existence of paired positive and energy modes. Since

$$(b_k^\dagger b_k)^2 = b_k^\dagger b_k , \quad (d_k^\dagger d_k)^2 = d_k^\dagger d_k \quad (14)$$

these have eigenvalues of only 0 or 1. Hence the familiar result in the vacuum sector that the charge operator has only integer eigenvalues.

### B. soliton sector

Let us repeat exactly the same steps in the soliton sector. Now we have to substitute for the Bose field operator in the Dirac Lagrangian the classical kink function in eq (6). The corresponding Dirac equation now becomes

$$(-i\alpha\partial_x + \beta m \tanh\frac{x}{\sqrt{2}})\psi(x) = E\psi(x) \quad (15)$$

For the two components  $\psi_{1,2}$  of the Dirac spinor, this yields the coupled equations

$$\begin{aligned} (-\partial_x + m \tanh\frac{x}{\sqrt{2}})\psi_2 &= E \psi_1 \\ (\partial_x + m \tanh\frac{x}{\sqrt{2}})\psi_1 &= E \psi_2 \end{aligned} \quad (16)$$

This equation will also have a set of positive energy solutions  $\eta_k(x)$  with some associated energy  $E_k$ . We need not find these solutions explicitly. But since the charge conjugation matrix  $\sigma_3$  again anticommutes with the Dirac hamiltonian, we know that for every positive energy solution  $\eta_k(x)$  there will exist a negative energy solution  $\tilde{\eta}_k(x)$  with energy  $-E_k$ . But now there is also an unpaired zero-energy solution

$$\eta_0 = \begin{pmatrix} A \exp\left(-m \int^x dy \tanh(y/\sqrt{2})\right) \\ 0 \end{pmatrix} \quad (17)$$

Such a normalisable solution to (16) exists because the soliton function  $\tanh\frac{x}{\sqrt{2}}$  which forms the background potential for the Dirac spinor tends to opposing limits  $\pm 1$  as  $x \rightarrow \pm\infty$ . In infinite spatial volume this solution has no partner. It is self charge conjugate :  $\sigma_3\eta_0 = \eta_0$ .

The mode expansion of the Fermi field operator now becomes

$$\Psi(x, t) = \sum_{k \neq 0} [b_k \eta_k(x) e^{-iE_k t} + d_k^\dagger \tilde{\eta}_k(x) e^{iE_k t}] + a \eta_0(x) \quad (18)$$

where  $a$  is the destruction operator for the zero-mode.

Unlike the vacuum sector built around  $\phi = 1$  which had a unique ground state ("the vacuum" in that sector), in the soliton sector there are two degenerate ground states because of the existence of the fermionic zero-mode. They are  $|sol\rangle$  and  $|\hat{sol}\rangle$  obeying

$$a|sol\rangle = b_k|sol\rangle = d_k|sol\rangle = 0 \quad (19)$$

and

$$|\hat{sol}\rangle \equiv a^\dagger|sol\rangle \ ; \ a|\hat{sol}\rangle = |sol\rangle \quad (20)$$

These are the two basic quantum soliton states of this system. They are energetically degenerate, but are distinguishable by their Charge.

$$\begin{aligned} Q &\equiv \frac{1}{2} \int dx [\Psi^\dagger(x, t), \Psi(x, t)] \\ &= \frac{1}{2} \sum_k \left( [b_k^\dagger, b_k] + [d_k, d_k^\dagger] \right) + 1/2 [a^\dagger, a] \\ &= \sum_k \left( (b_k^\dagger b_k - 1/2) - (d_k^\dagger d_k - 1/2) \right) + (a^\dagger a - 1/2) \\ &= \sum_k \left( b_k^\dagger b_k - d_k^\dagger d_k \right) + a^\dagger a - 1/2 \end{aligned} \quad (21)$$

Notice that the piece  $(-1/2)$  coming from the zero- mode commutator remains uncanceled because it does not have a charge conjugate partner. Since the operators  $b_k^\dagger b_k$ ,  $d_k^\dagger d_k$  and  $a^\dagger a$  all have eigenvalues of 0 or 1, it then follows that the total Charge (Number) operator  $Q$  has half-integral eigenvalues. It should be emphasized that eq(21) is an operator equation for  $Q$ . The half-integer appearing in it will be reflected in its eigenvalues and not just its expectation values. In particular, the two degenerate soliton states have eigenvalues of  $\pm 1/2$  respectively for the total number operator  $Q$  :

$$Q |sol\rangle = -(1/2)|sol\rangle \quad Q |\hat{sol}\rangle = (1/2)|\hat{sol}\rangle \quad (22)$$



This, in its barest form, is the prototype example of the Jackiw-Rebbi discovery that in some field theories there can be states carrying fractional eigenvalues of the Number operator. For a fuller discussion including similar results in three-dimensional models see their original paper [1].

### III. POLYACETYLENE

This discovery by Jackiw and Rebbi was clearly very remarkable. But it is not easy to "understand" the result physically in terms of our familiar intuition with quantum field theory. How can the Number Operator which is widely used to count the number of particles in QED and so many other theories, and which even in the Higgs model used by Jackiw and Rebbi has only integer eigenvalues in the vacuum sector, yield fractional values in another sector ? Yet the proof given is so simple and transparent that, stare at it as we may, we have no choice but to accept the result within the parameters of its derivation. In my own attempts to make peace with the result (before embarking on the more detailed study with John), I loosely attributed it to the vagaries of the infinite degrees of freedom of continuum field theory. [ We know from high school that formally summing divergent series of integers can yield fractions.] But clearly some further clarification of this phenomenon was called for.

The need for clarification became more compelling when a similar result was derived not in a model field theory, but in a down-to-earth experimentally accessible polymer system by Su Schrieffer and Heeger [2]. Three years after the Jackiw-Rebbi work, but quite independently they showed that the same phenomenon of fermion number  $1/2$  can occur for soliton states in the long chain molecule *trans*-Polyacetylene.

That the same phenomenon occurs in Polyacetylene (we will henceforth drop the prefix "trans") as in the Jackiw-Rebbi field theory is not accidental. As pointed out by Jackiw and Schrieffer [4] , the former system has the same structure in the continuum limit as the field theory model. Polyacetylene is the molecule  $(CH)_n$  with large  $n$ , where the Carbon ions

form a long chain with the H atoms sticking out transverse to the chain. If we consider the flow of electrons along the chain, the system can be viewed as consisting of electrons and bosons (the phonons of lattice vibration of the Carbon ions along the chain) in one space dimension just as in the field theory model .

The Hamiltonian for the electron-lattice system can be taken, for each of the two spin states of the electron and for small ion displacements as (see [4]) :

$$H = \sum_n \left[ \left( \frac{p_n^2}{2\mu} + K/2 (u_{n+1} - u_n)^2 \right) + \left( D_{n+1}^\dagger D_n + h.c. \right) \left( u_{n+1} - u_n - 1/(2a) \right) \right] \quad (23)$$

where  $n$  labels lattice sites,  $p_n$  and  $u_n$  are ion momenta and displacements respectively ,  $D_n$  is electron destruction operator and  $K$  and  $a$  are constants. All other constants have been absorbed into definitions for simplicity.

Our primary interest is in the fermions . As far as the lattice vibrations go, let us just state the Su *et al* result [2] that the lattice system undergoes dimerisation doubling its spatial period (a Peierls transition) . There are two degenerate ground states. In one , the mean value of the displacement  $u_n$  instead of being zero takes the staggered value of

$$\phi_n \equiv 4(-1)^n u_n = 1 \quad (24)$$

for all  $n$ . In the other

$$\phi_n = -1 \quad (25)$$

The two degenerate ground states further provide the possibility of domain wall (soliton) configurations connecting the two phases , i.e.  $\phi_n \rightarrow \pm 1$  (or  $\mp 1$ ), as  $n \rightarrow \pm\infty$  respectively. Thus the boson subsystem has four sectors of states – two of them being soliton sectors – just as in the field theory discussed in the previous section.

Furthermore, the boson coordinates  $u_n$  again act as the background potential for the electrons . In fact the electronic part of the hamiltonian (23) becomes in the continuum approximation exactly the same as that of the Dirac field theory. John and I , as part of

our work on these systems [9], offered a simple derivation showing that the polyacetylene hamiltonian is just a realisation of the Kogut-Susskind lattice regularisation of the Dirac system [6]. Let me sketch that derivation. The electronic part of (23) is

$$H_{elec} = \sum_n \left[ D_{n+1}^\dagger D_n (u_{n+1} - u_n - 1/(2a)) \right] + \text{herm.conj} \quad (26)$$

Define staggered variables

$$B_{2r-1} \equiv (-1)^r D_{2r-1} ; C_{2r} \equiv (-1)^r D_{2r} \quad (27)$$

with  $\phi_n \equiv 4(-1)^n u_n$  for all  $n$  as already defined above. Then,

$$\begin{aligned} H_{elec} &= \sum_r \left[ \left( D_{2r+1}^\dagger D_{2r} (u_{2r+1} - u_{2r} - 1/2a) + D_{2r}^\dagger D_{2r-1} (u_{2r} - u_{2r-1} - 1/2a) \right) + h.c. \right] \\ &= (1/4) \sum_r \left[ B_{2r+1}^\dagger C_{2r} (\phi_{2r+1} + \phi_{2r} + 2/a) + C_{2r}^\dagger B_{2r-1} (\phi_{2r} - \phi_{2r-1} - 2/a) \right] + h.c. \\ &= \sum_r \left( \frac{B_{2r+1}^\dagger - B_{2r-1}^\dagger}{2a} \right) C_{2r} + (1/4) B_{2r+1}^\dagger C_{2r} (\phi_{2r+1} + \phi_{2r}) + (1/4) B_{2r-1}^\dagger C_{2r} (\phi_{2r} + \phi_{2r-1}) \\ &+ h.c. \end{aligned} \quad (28)$$

In the continuum limit, as  $a \rightarrow 0$  and  $\sum_r \rightarrow \int dx$ , this reduces to

$$\begin{aligned} H_{elec} &= \int dx \left( \frac{\partial B^\dagger}{\partial x} C + C^\dagger \frac{\partial B}{\partial x} \right) + \phi(x) (B^\dagger C + C^\dagger B) \\ &= \int dx \left( - B^\dagger \frac{\partial C}{\partial x} + C^\dagger \frac{\partial B}{\partial x} \right) + \phi(x) (B^\dagger C + C^\dagger B) \end{aligned} \quad (29)$$

This is just the Dirac Hamiltonian with Yukawa coupling :

$$H_F = \Psi^\dagger \left( -i\sigma_2 \frac{\partial}{dx} + \sigma_1 \phi(x) \right) \Psi \quad (30)$$

with the upper and lower components of the 2-spinor being identified respectively with the odd-and even- site electron operators :

$$\Psi(x) \equiv \begin{pmatrix} B(x) \\ C(x) \end{pmatrix} \quad (31)$$

The charge operator operator at each site also has the same commutator form as in the continuum theory's charge density :

$$\rho_n = D_n^\dagger D_n - (1/2) = (1/2) [D_n^\dagger, D_n] \quad (32)$$

where the  $1/2$  subtracted at each site can be attributed to the neutralizing background (ionic) charge *per spin state*.

Given this mapping from the polyacetylene system to the field theory model used by Jackiw and Rebbi, it is not surprising that fractional charged states arise in the former system too.

But in the context of a real polymer the result becomes still more mysterious. A molecule of Polyacetylene has after all some finite number of electrons and it is hard to imagine how, even if it twists itself into some topological soliton state, it could have a fractional number of electrons. The need to understand this better became all the more compelling because Su, Schrieffer- and Heeger referred to experimental signals supporting the result. See for instance ref ([7]). [ Because of the presence of the two spin degrees of freedom in actual polyacetylene, as distinct from a truly 1+1 dimensional model, the fractional charge gets doubled and becomes integral; but a signature of the effect can still be observed in the form of an unfamiliar charge-spin combination of excitations. There will be neutral excitations with spin and charged excitation which are spinless.]

It should be emphasised that an expectation value of one-half would cause no surprise. Obviously that can easily arise for an operator with integral eigenvalues. For instance if we take any particle in the ground state of a double-well potential and look for it in one of the wells, you will find it there half the times and not find it there the other half, giving an expectation value of  $1/2$ . While in the Su et al paper, fractional values were not claimed as eigenvalues, in the prototype field theoretic case we have explicitly seen in sec.II that the eigenvalues themselves are half-integers. This is what puzzled me and I was fortunate in having John Bell join me in worrying about it further.

#### IV. EIGENVALUE OR EXPECTATION VALUE ?

Now let me summarize what John and I did in our attempt to understand better what these half integral values of the number operator mean [8], [9] . [ We later learnt that Kivelson and Schrieffer had also been independently investigating the eigenvalue status of fractional charge around the same time with essentially a similar resolution of the issues [10]. See also the work of Jackiw *etal* which provided further clarification [11]. I will describe here only the work done with John Bell.] Our strategy was to treat the infrared and ultraviolet limits of the problem more carefully. In one paper [8], we looked at the continuum field theory used by Jackiw and Rebbi, but in a finite volume, which could be later taken to infinity. In a subsequent paper we considered what would happen if we put an ultraviolet cut-off as well, by going to the finite polyacetylene chain [9].

Let me begin with the field theory example of Jackiw and Rebbi which was discussed in section II. The result derived there that the total charge operator has half-integral eigenvalues cannot be disputed in the infinite spatial volume case ( $L \rightarrow \infty$  in the one dimensional field theory model considered there). However suppose we kept the spatial volume  $2L$  finite to start with, and later go the  $L \rightarrow \infty$  limit [8]. In particular , in the soliton sector, we go back and solve the same Dirac equation

$$\begin{aligned} (-\partial_x + m \tanh \frac{x}{\sqrt{2}}) \psi_2 &= E \psi_1 \\ (\partial_x + m \tanh \frac{x}{\sqrt{2}}) \psi_1 &= E \psi_2 \end{aligned} \tag{33}$$

but in a finite volume ( $-L \leq x \leq L$ ) with appropriate boundary conditions on the spinor  $\psi_\alpha$  at  $x = \pm L$ . The natural boundary conditions which leave the Dirac hamiltonian hermitian are  $\psi_{1,2}(-L) = \psi_{1,2}(L)$ . Once again every positive energy solution  $\eta_k$  will have a negative energy partner  $\tilde{\eta}_k$ . We do not need their details. But now there are two zero energy solutions :

$$\eta_0(x) = \begin{pmatrix} A \exp\left(-m \int_0^x dy \tanh(y/\sqrt{2})\right) \\ 0 \end{pmatrix} \tag{34}$$

and

$$\tilde{\eta}_0(x) = \begin{pmatrix} 0 \\ A \exp\left(-mL + m \int_0^x dy \tanh(y/\sqrt{2})\right) \end{pmatrix} \quad (35)$$

While the solution (34) is the same one as before, localised around the origin, the second solution (35) has support near the edges  $\pm L$ . In section II where we started with an infinite volume problem, this second solution could not be entertained. But for any finite  $L$  however large this second solution, finite everywhere and normalisable, is certainly present. Each of these two zero-modes is charge self-conjugate and both will appear in the expansion of the Dirac field operator, along with the non-zero energy modes :

$$\Psi(x, t) = \sum_{k \neq 0} [b_k \eta_k(x) e^{-iE_k t} + d_k^\dagger \tilde{\eta}_k(x) e^{iE_k t}] + a \eta_0(x) + c^\dagger \tilde{\eta}_0(x) \quad (36)$$

Correspondingly there will now be four degenerate ground states in the soliton sector, namely,  $|sol\rangle$  defined by

$$b_k |sol\rangle = d_k |sol\rangle = a |sol\rangle = c |sol\rangle = 0 \quad (37)$$

and  $|\widetilde{sol}\rangle = a^\dagger |sol\rangle$ ,  $|\overline{sol}\rangle = c^\dagger |sol\rangle$ , and  $|\widetilde{sol}'\rangle = a^\dagger c^\dagger |sol\rangle$ .

The expansion (36) when inserted into the definition of the charge operator

$$Q \equiv \frac{1}{2} \int_{-L}^L dx \left[ \Psi^\dagger(x, t), \Psi(x, t) \right] \quad (38)$$

will yield

$$Q = \sum_k \left( (b_k^\dagger b_k - 1/2) - (d_k^\dagger d_k - 1/2) \right) + (a^\dagger a - 1/2) + (c^\dagger c - 1/2) \quad (39)$$

We see that there are now no unpaired terms of  $1/2$  and the total charge operator has only integral eigenvalues. One can now let the volume  $2L$  become larger and larger and at each stage both zero modes will be present and the total charge will have only integral eigenvalues.

That one can restore integral valued charges by this strategy of starting with a finite volume and then letting  $2L \rightarrow \infty$  can also be understood in a different way. Note that if

we had closed our line ( $-L \leq x \leq L$ ) by identifying the points  $x = \pm L$ , that is equivalent to putting the system on a circle with a soliton at  $x=0$  and an antisoliton at  $x = L$ . This in turn amounts to working in the vacuum sector, where we saw that the charge is integral.

While integral charge has been thus restored, if we were to stop here we would have lost all the important physics unearthed by Jackiw and Rebbi. Notice that while there are now two zero modes, both localised with width of order  $1/m$ , one of them,  $\tilde{\eta}_0(x)$  given in eq.(35) is stuck at the edges of the sample. But the other zero mode  $\eta_0(x)$  in (34), is in the middle and can move about as an excitation, carrying charge  $1/2$  in some sense that needs to be made more precise.

To pick just the charge carried by this middle zero-mode, we can try and define a "partial charge operator" :

$$P_l \equiv \int_{-l}^l dx \rho(x) \quad (40)$$

where  $l, L \rightarrow \infty$  in the order  $L \gg l \gg 1/m$ . In this limit the partial charge  $P_l$  clearly includes the central zero mode while excluding the second zero mode at the edges  $\pm L$ . However, the soliton ground state will not be an eigenstate of  $P_l$ , but only of the total charge operator  $Q$  in eq(39). The charge density operator  $\rho(x)$  given in eq(12) and partial charge operators such as  $P_l$  will excite particle-hole pairs when acting on the soliton state  $|sol\rangle$ . But consider expectation values in this soliton state.

$$\begin{aligned} \langle sol | \rho(x) | sol \rangle &= (-1/2) \sum_{k \neq 0} \left[ \eta_k^*(x) \eta_k(x) - \tilde{\eta}_k^*(x) \tilde{\eta}_k(x) \right] \\ &+ (1/2) \left( \tilde{\eta}_0(x) \tilde{\eta}_0(x) - \eta_0(x) \eta_0(x) \right) \\ &= (1/2) \left( \tilde{\eta}_0(x) \tilde{\eta}_0(x) - \eta_0(x) \eta_0(x) \right) \end{aligned} \quad (41)$$

where in the last line, contributions from non-zero modes cancel for each  $k$  by charge conjugation. Hence

$$\begin{aligned} \langle sol | P_l | sol \rangle &= \int_{-l}^l dx (1/2) \left( \tilde{\eta}_0^2(x) - \eta_0^2(x) \right) \\ &= -(1/2) \int_{-l}^l dx \eta_0^2(x) = -(1/2) \end{aligned} \quad (42)$$

Because this is not an eigenstate of  $P_l$  there will be fluctuations. Since

$$(P_l - \langle P_l \rangle) |sol\rangle = \sum_{k,k'} \int_l^l dx (\eta_k(x) \tilde{\eta}_{k'}(x)) |k, k'\rangle \quad (43)$$

We have

$$\langle sol | (P_l - \langle P_l \rangle)^2 |sol\rangle = \sum_{k,k'} \left| \int_l^l dx \eta_k(x) \tilde{\eta}_{k'}(x) \right|^2 \quad (44)$$

The integrand is positive and hence the fluctuations are non-zero. In fact they diverge logarithmically.

Such large fluctuations in partial charges are not special to soliton states. They occur even in familiar vacuum states and are because of the sharp boundary of the defining domain. One can kill the fluctuations while still retaining a value of 1/2 by defining a more sophisticated partial charge operator, one with fuzzy edges. Let

$$\tilde{P}_{l,d} \equiv \frac{1}{d} \int_l^{l+d} dl' P_{l'} \quad (45)$$

When  $L, l, d \rightarrow \infty$  with  $L \gg l, d$  then once again,

$$\langle sol | \tilde{P}_{l,d} |sol\rangle = -(1/2) \quad (46)$$

To obtain the fluctuations of this operator note that  $[\rho(x) - \langle \rho(x) \rangle]$  connects the state  $|sol\rangle$  to the particle-hole states  $|k, k'\rangle$ . We therefore have

$$\begin{aligned} \langle k, k' | P_l - \langle P_l \rangle |sol\rangle &= \langle k, k' | \int_{-l}^l dx (\rho - \langle \rho \rangle) |sol\rangle \\ &= \frac{1}{i(E_k + E_{k'})} \langle k, k' | \int_{-l}^l dx \frac{\partial \rho}{\partial t} |sol\rangle \\ &= \frac{1}{i(E_k + E_{k'})} \langle k, k' | (j(l) - j(-l)) |sol\rangle \end{aligned} \quad (47)$$

by current conservation. Note that the current  $j$  has matrix elements  $\langle k, k' | j(x) |sol\rangle = \eta_k^*(x) i\alpha \tilde{\eta}_{k'}(x)$ . Hence,

$$\langle k, k' | \tilde{P}_{l,d} - \langle \tilde{P}_{l,d} \rangle |sol\rangle = \frac{1}{d} \int_l^{l+d} dl' \frac{1}{i(E_k + E_{k'})} \left[ \eta_k^* i\alpha \tilde{\eta}_{k'} \right]_{-l'}^{l'} \quad (48)$$



To obtain the exact result for this one must know analytic expressions for all the non-zero energy modes  $\eta_k(x)$  and  $\tilde{\eta}_k(x)$ . We had in fact obtained these solutions for the simplified case where the soliton function is taken as a step function  $\phi(x) = \Theta(x)$  instead of  $\tanh x$ . This simplification does not affect the issues of interest to us. But even quite generally one can see that for large  $l$  far away from the soliton center ( $l \gg 1/m$ ) the solutions  $\eta_k(l)$  will be trigonometric functions of  $kl$ . Since  $E_k \approx k$  for large  $k$ , the matrix element in (47) will behave as  $1/k$  for large  $k$  and give rise to the logarithmically divergent fluctuations in eq. (44). But the charge operator with fuzzy edges  $\tilde{P}_{l,d}$  defined in (45) has an additional integral over  $dl'$  which will bring down an additional convergence factor of  $1/k$ . There is also the extra factor  $1/d$  in the definition which goes to zero. Altogether

$$\begin{aligned} \langle sol | (\tilde{P}_{l,d} - \langle \tilde{P}_{l,d} \rangle)^2 | sol \rangle &= \sum_{k,k'} |\langle k, k' | \tilde{P}_{l,d} - \langle \tilde{P}_{l,d} \rangle | sol \rangle|^2 \\ &\approx \frac{1}{d^2} \sum_{k,k'} f(k, k') \end{aligned} \quad (49)$$

where the function  $f(k, k')$  has a convergent sum over  $k, k'$ . Clearly this vanishes as  $d \rightarrow \infty$ . In other words, one can define a suitable partial charge operator with fuzzy edges for which the soliton state has a value of  $-1/2$  with *no* fluctuations! This fractional charge then *is* an eigenvalue.

It is very possible that if one were to measure the charge in the neighborhood of the soliton's center, one is using some such partial charge operator with fuzzy edges. In that case the measured value of  $1/2$  can be promoted to the status of an eigenvalue. Further, since these fractions are eigenvalues of complicated partial charge operators and not of the total charge (Number) operator  $Q$ , there is no conceptual problem reconciling the result with our intuition about the latter.

### A. Polyacetylene re-visited

Let us describe briefly what a corresponding analysis [9] yields for polyacetylene which is a lattice chain of some finite number  $N$  of sites amongst which the electrons hop. I will

emphasize just the additional insights that emerge here due to the fact that it is an even better regulated system than the field theory in finite volume that we just studied. It has only a finite number of degrees of freedom and hence no ultraviolet or infrared problems. The lattice Dirac Hamiltonian in eq(28) yields the following coupled equations for the electron wave functions  $b_n$  and  $c_n$  at odd and even sites respectively :

$$\begin{aligned} E b_n &= (1/4)c_{n+1} (\phi_{n+1} + \phi_n - 2/a) + (1/4)c_{n-1} (\phi_n + \phi_{n-1} + 2/a) \\ E c_n &= (1/4)b_{n+1} (\phi_{n+1} + \phi_n + 2/a) + (1/4)b_{n-1} (\phi_n + \phi_{n-1} - 2/a) \end{aligned} \quad (50)$$

Clearly, given a solution for some positive energy  $E$  we can get another solution of negative energy ( $-E$ ) by replacing  $b_n \rightarrow -b_n$ ,  $c_n \rightarrow c_n$ ,  $E \rightarrow -E$  in the above equations. Thus positive and negative energy solutions again come in pairs. But suppose the lattice had altogether only an odd number  $N$  of sites. Then the Dirac operator can have only the same odd number of independent solutions. Therefore there must be one or more zero-energy solutions.

Notice that this argument for the existence of zero modes has nothing to do with whether the background boson field is in the topological soliton sector or the normal vacuum sector. All you need is an odd number of lattice sites ! One can explicitly verify this result. Take  $N = 2M + 1$  with, say,  $M$  odd. A zero energy solution of the lattice Dirac equation (50) clearly exists in which the even site variable  $c_n = 0$  for all  $n$  while the odd site variables  $b_n$  satisfy

$$\frac{b_{n+1}}{b_{n-1}} = \frac{2 - a(\phi_n + \phi_{n-1})}{2 + a(\phi_n + \phi_{n+1})} \quad (51)$$

This will be true for any background Bose field  $\phi_n$ . In the uniform phase (the "vacuum sector here)  $\phi_n = 1$  for all  $n$ . Hence the solution is

$$b_n = b_0 \exp(-Kan) \quad ; \quad c_n = 0 \quad (52)$$

where the constant  $K$  obeys  $\tanh(Ka) = a$ . If  $\phi_n$  is a soliton, say having the simple form  $\phi_n = \frac{n}{|n|}$  then the fermion zero mode is

$$b_n = b_0 \exp(-Ka|n|) \quad ; \quad c_n = 0 \quad (53)$$

Given the single zero mode in both sectors, one can show that the ground states in both sectors will have total charge of  $(-1/2)$  as eigenvalues. By simply adding  $+ 1/2$  to the definition of charge one can trivially make eigenvalues integral in both sectors as was the case in the finite-volume field theory. Similarly, if the number of lattice points  $N$  had been even, there would have been either no zero modes or an even number of them in both sectors, leading to integral total charge.

Thus the important lesson we learn is that as far as the **total** charge is concerned, there is no half integral eigenvalue in the soliton sector *as compared* to the uniform phase. This is true in finite chains even if there is a single isolated zero mode in the soliton sector, because a zero mode will then exist in the vacuum sector too.

Of course one can also define for lattice chains partial charges that pick up the zero mode in the soliton sector but not the one in the vacuum sector. [ Recall in the example mentioned that the zero mode in the soliton sector given in (52) is located near the center whereas in the vacuum sector even though a zero mode (53) was present it was near one of the edges.] As happened in the field theory case discussed earlier in this section, such partial charge operators would have a half-integral value in the soliton state. This value would get promoted to the status of an eigenvalue with no fluctuations if, in the limit of a very long chain, the operator covered a region with suitably fuzzy edges (see ref [9] for more details).

## V. CONCLUSION

I have tried to outline in the last section the work that John Bell and I had done on Fractional charge, preceded in earlier sections by some background on where matters stood when we began our work. I believe our work had thrown light on which operators do have eigenvalues of  $1/2$  (as distinct from expectation values ) and which don't. In the process we had also laid to rest the concern that had motivated us to study this problem — which was that the total number operator of a physical system should not have fractional eigenvalues.

That would have made no sense. We showed that the total charge operator defined to have integral eigenvalues in the absence of solitons will, in a well regulated theory, continue to have integer eigenvalues even in the presence of solitons. But suitable localised partial charge operators can have fractional eigenvalues and fractional charge should be interpreted as corresponding such an operator. Indeed these may well be what charge measurements in experiments employ. As mentioned already, Kivelson and Schrieffer [10] had come to the same conclusion independently.

The work discussed above is nearly 20 years old. This article is primarily devoted to John Bell and his work , so this is not the place to discuss in any detail other developments on this topic around that time or since then. However, for the sake of completeness let me summarize in a few sentences some of these developments .

Soon after charge  $1/2$  states were discovered in field theory and polyacetylene, the possibilities of charge at other fractions were unearthed both in polymer physics and in model field theories ( see for example [12] and [13]) .

A major addition to the list of fractionally charged objects was the theoretical discovery by Laughlin in an entirely new arena. He showed that quasi-particles in fractional quantum Hall (FQH) systems carry fractional charge [14]. Subsequently these have also been experimentally observed in "shot-noise" experiments [15]. To the best of my knowledge the quantum mechanical status of this quasi-particle charge in FQH has not been analyzed as carefully as in the case of polyacetylene or field theory models. The former have been identified through ingenious but indirect arguments involving plasma analogies and Aharanov-Bohm/ Berry phases rather than by an explicit analysis of the charge operators and their eigenstates. The FQH quasi-particles are more difficult states to study. They correspond to genuinely correlated many-body states as distinct from the polyacetylene and Jackiw-Rebbi soliton states considered above, which could be analysed in terms of single particle solutions of the Dirac equation. But broadly speaking, the status of the fractional charge in FQH is believed to be similar to those discussed above. It is localized near the center of the quasi-particle wavefunction and the complementary missing fraction is believed

to be near the edge of the Hall sample.

Finally, very recently, it has been claimed by Maris [16] that some "bubbles" formed in liquid helium which contain a trapped electron can then fission into daughter bubbles each of which carries a fragment of the original electron . However, in an analysis of this phenomenon, the pioneers of fractional charge Jackiw, Rebbi and Schrieffer have argued that these fractions in helium bubbles are just expectation values of the sort familiar in local charge measurements in any double well quantum system [17]. This is an ongoing area of work.

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