Majorana zero modes and their generalizations in condensed matter

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מכון ויצמן למדע
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Plan

- A few words about topological order
- p-wave superconductors
- Kitaev’s chain
- Experimental platforms and physical signatures
- New platform: topological superconductivity in planar Josephson junctions
- Beyond Majoranas

Topological states of matter

- Gapped states of matter, do not break any symmetry
- Cannot be deformed adiabatically to "trivial" (atomic) insulator: phase transition *must occur* along the way
- "Hidden" (non-local, or "topological") order in the ground state wavefunction

Examples:

- Topological insulators (2D and 3D), Haldane S=1 chain: "Symmtry Protected Topological phases" (SPT), distinct from the trivial phase so long as symmetry is maintained
- Quantum Hall effect

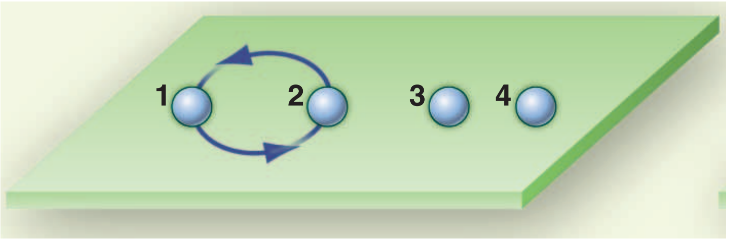
Topologically ordered states of matter (e.g. fractional QH in D=2):

- Point-like excitations with fractionalized statistics (anyons!), sometimes with fractional quantum numbers (e.g. charge)
- Ground state degeneracy depends on topology (genus) of manifold

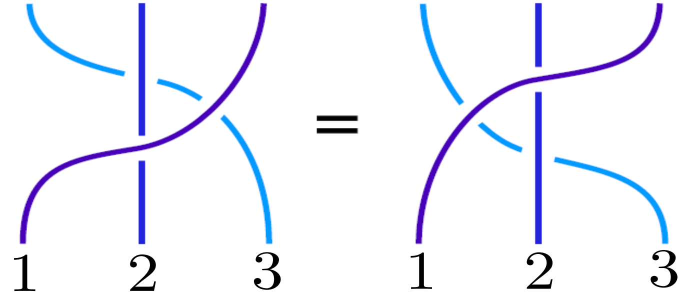
Non-Abelian statistics

- Topologically ordered states can exhibit *Non-Abelian* statistics
- In the presence of excitations ("quasi-particles"), ground state is *multiply degenerate*
- Moving excitations around each other ("braiding") implements unitary transformation that depends on *topology*, not geometry, of path
- Interactions necessary (true for all topologically ordered states)

$$|\psi_i\rangle \rightarrow \sum_j U_{ij} |\psi_j\rangle$$



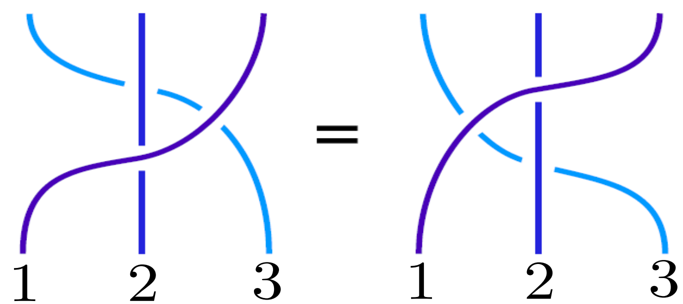
Braid group:



This talk: non-Abelian properties of zero modes at edges and defects

- Zero modes that appear at edges/defects (e.g., vortices) of certain kinds of superconductors
- Do not require topological order; no dependence of g.s. degeneracy on topology of manifold; can appear without interactions (or interactions treated within mean-field theory)
- Do nevertheless support robust g.s. degeneracy (not symmetry protected) and non-Abelian statistics

Braid group:



BdG formalism and p-wave superconductors

$$H = \underbrace{\sum_{i,j} -t_{ij}\,\psi_i^\dagger \psi_j}_{\text{hopping}} \; \underbrace{- \mu \sum_i \psi_i^\dagger \psi_i}_{\text{chemical potential}} + \underbrace{\sum_{i,j} \Delta_{ij}\,\psi_i^\dagger \psi_j^\dagger}_{\text{pairing potential}} + h.c.$$

$$= \underbrace{(\psi_1^\dagger \ldots \psi_N^\dagger \quad \psi_1 \ldots \psi_N)}_{\Psi^\dagger} \underbrace{\begin{pmatrix} h_{ij} & \Delta_{ij} \\ (\Delta^\dagger)_{ij} & -h_{ij}^* \end{pmatrix}}_{\mathcal{H}:\ 2N \times 2N} \underbrace{\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \\ \psi_1^\dagger \\ \vdots \\ \psi_N^\dagger \end{pmatrix}}_{\Psi}$$

Nambu Spinor (Here $h_{ij} = \frac{1}{2}(-t_{ij} - \mu\,\delta_{ij})$)

Quadratic in ψ's $\rightarrow$ can be solved!

BdG formalism and p-wave superconductors

Bugoliubov transformation:

$$\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \\ \psi_1^\dagger \\ \vdots \\ \psi_N^\dagger \end{pmatrix} = \underset{\text{Unitary}}{U} \underbrace{\begin{pmatrix} f_1 \\ \vdots \\ f_N \\ f_1^\dagger \\ \vdots \\ f_N^\dagger \end{pmatrix}}_{F}$$

Check:

$$\{f_i^\dagger, f_j\} = \delta_{ij}$$

$$\mathcal{H} = \Psi^\dagger H \Psi = F^\dagger (U^\dagger H U) F$$

Choose U to diagonalize H:

$$U^\dagger H U = \begin{pmatrix} E_1 & & \\ & \ddots & \\ & & E_{2N} \end{pmatrix}$$

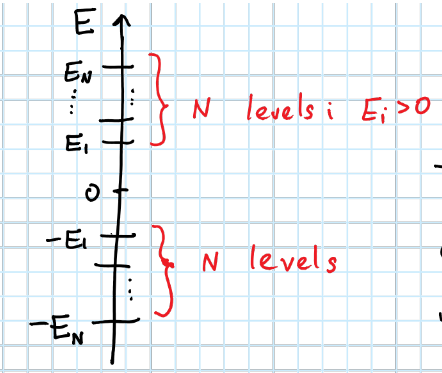
Symmetry of the BdG Hamiltonian:

$$C^{-1} H C = -H, \quad \text{where} \quad C = \begin{pmatrix} 0 & \mathbb{1}_{N\times N} \\ \mathbb{1}_{N\times N} & 0 \end{pmatrix} K$$

"Particle - hole" symmetry

K: Complex conjugation

BdG formalism and p-wave superconductors



The levels at E_i and $-E_i$ are not independent: they are "two halves of the same state".

After the transformation:

$$\mathcal{H} = F^{+} \begin{pmatrix} E_1 & & & \\ & E_N & & \\ & & -E_1 & \\ & & & -E_N \end{pmatrix} F$$

$$= \sum_{i=1}^{N} E_i \left(f_i^{+} f_i - f_i f_i^{+} \right)$$

$$= \sum_{i=1}^{N} E_i \left(2 f_i^{+} f_i - 1 \right)$$

Many-body ground state: $f_i^{+} f_i |0\rangle = 0 \quad \forall \; i = 1 \dots N$

(Since $E_i > 0$)

Excited states: $\prod_{j=1}^{n} f_{\alpha_1}^{+} \dots f_{\alpha_n}^{+} |0\rangle$

Energy (rel. to ground state

$2(E_{\alpha_1} + \dots + E_{\alpha_n})$

1D p-wave superconductor

$$H = \sum_j \left(-t c_i^\dagger c_{i+1} + H.c. - \mu c_i^\dagger c_i + \Delta c_i^\dagger c_{i+1}^\dagger + H.c. \right)$$

Fourier transform:

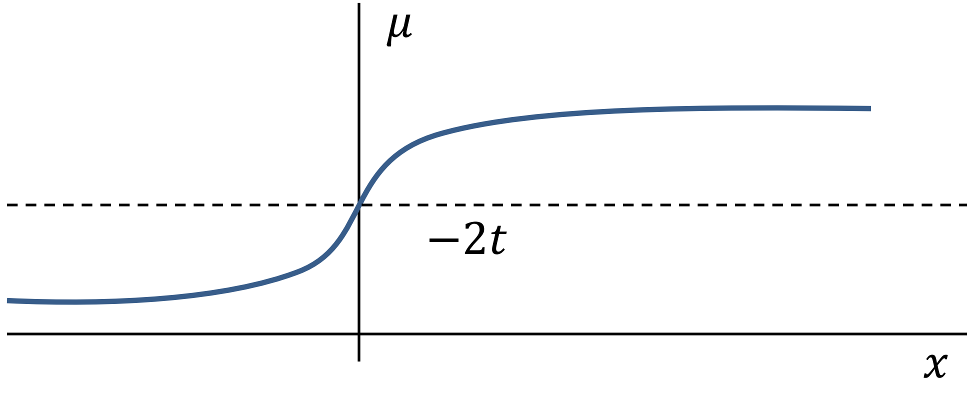
$$H = \int dk \, (-2t \cos k - \mu) \, c_k^\dagger c_k + \Delta \sin k c_k^\dagger c_{-k}^\dagger + h.c.$$

Spectrum:

$$E_k = \pm \sqrt{(-2t \cos k - \mu)^2 + |\Delta|^2 \sin^2 k}.$$

Phase transition at $\mu = \pm 2t$, at $k = 0$, $k = \pi$, respectively.

Change μ slowly in space



1D p-wave superconductor

$$H = \int dk\,(-2t\cos k - \mu)\,c_k^\dagger c_k + \Delta \sin k_x c_k^\dagger c_{-k}^\dagger + h.c.$$

For small k, expand to first order in k, and go back to real space:

$$H \approx \begin{pmatrix} -\delta\mu(x) & -i\Delta\partial_x \\ -i\Delta\partial_x & \delta\mu(x) \end{pmatrix}$$

$$H = -\delta\mu(x)\tau^z - \Delta\tau^x i\partial_x.$$

Look for zero energy solution:

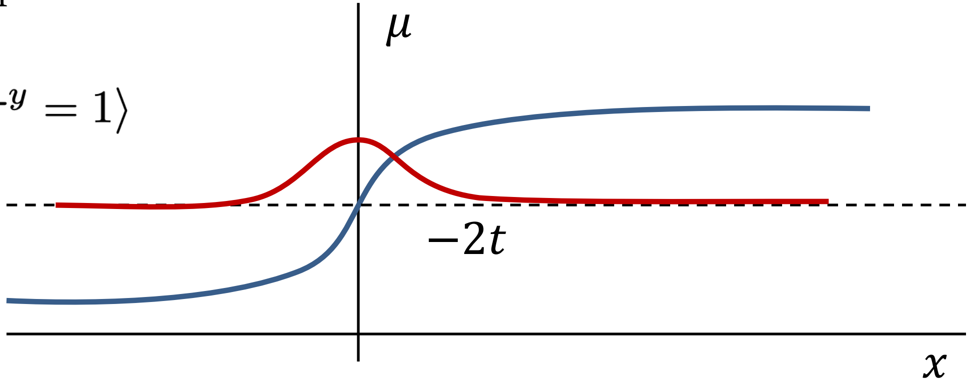
$$\left[-\delta\mu(x)\tau^z - i\Delta\tau^x\partial_x\right]\psi = 0$$

Multiply by $i\tau^x$:

$$\left[\delta\mu(x)\tau^y + \Delta\partial_x\right]\psi = 0$$

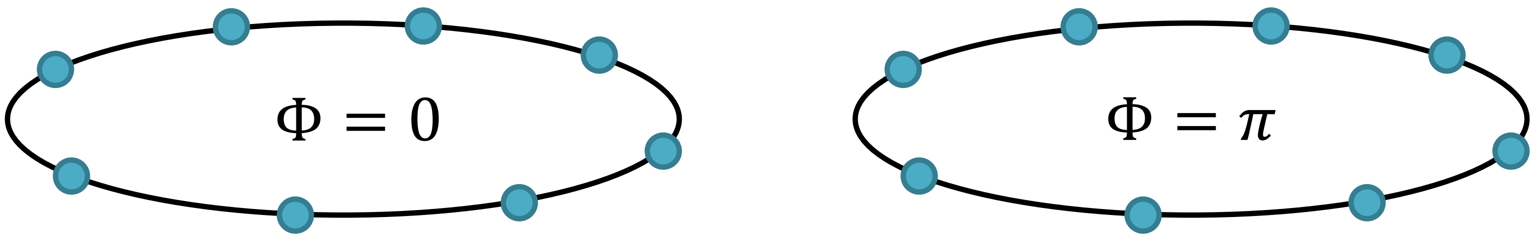
Jackiw-Rebbi soliton

$$\psi = e^{-\int_0^x dx' \frac{\delta\mu(x)}{\Delta}}|\tau^y = 1\rangle$$

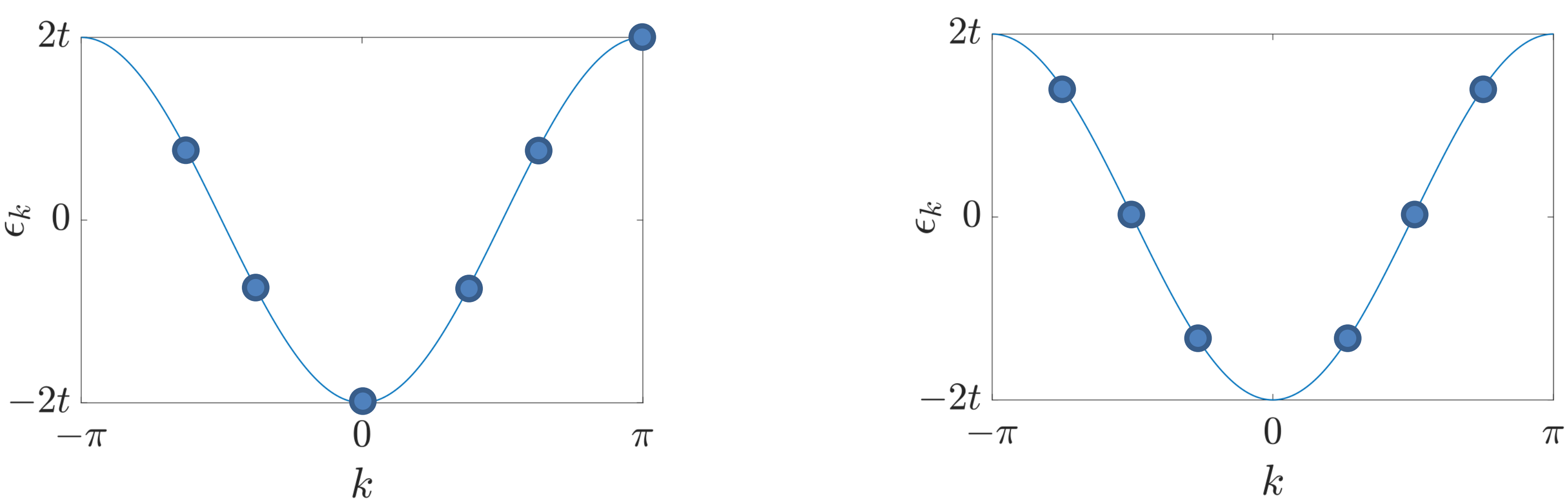


1D p-wave superconductor

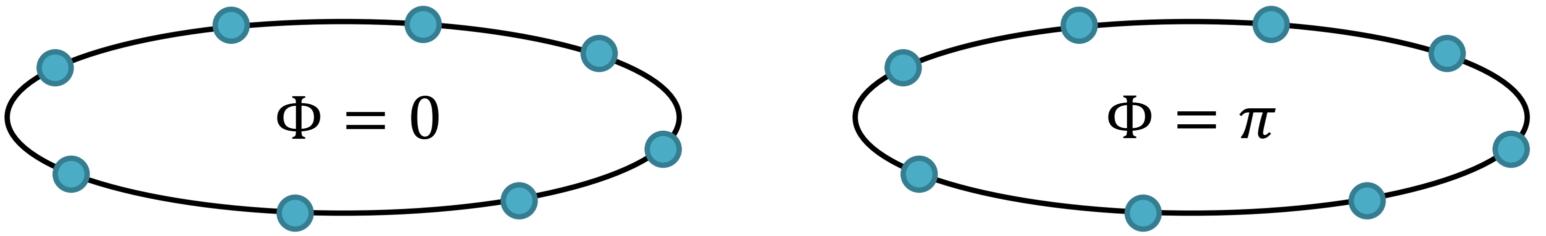
- We would like to identify $|\mu| < 2t$ and $|\mu| > 2t$ as two different phases. But what distinguishes the two phases?
- In the "topological" phase ($|\mu| < 2t$):



Fermion parity of the ground state is opposite!



1D p-wave superconductor



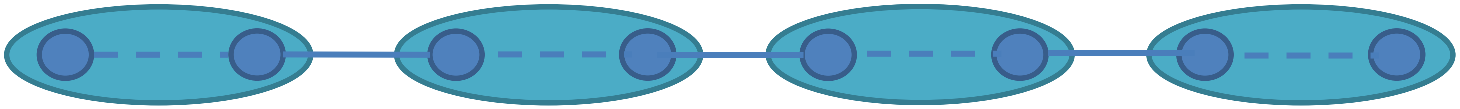
Topological phase: fermion parities of ground states with periodic and anti-periodic b.c. are opposite.

Consequences:

- Trivial and topological phases must be separated by gap closing (fermion parity in one sector must switch)
- In a non-interacting translationally invariant system, topological phase transition is characterized by gap closing either at $k = 0$ or $k = \pi$

Another derivation of the existence of zero modes (Kitaev)

$$H = \sum_j -tc_j^\dagger c_{j+1} + \Delta c_j^\dagger c_{j+1}^\dagger + H.c. - \mu c_j^\dagger c_j$$



Write in terms of Majorana operators: $c_j = \dfrac{\alpha_j + i\beta_j}{2}$ $\qquad \alpha_j = c_j + c_j^\dagger$, $\beta_j = c_j - ic_j^\dagger$

$$\alpha^\dagger = \alpha,\ \beta^\dagger = \beta,\ \{\alpha_i, \beta_j\} = \delta_{ij}$$

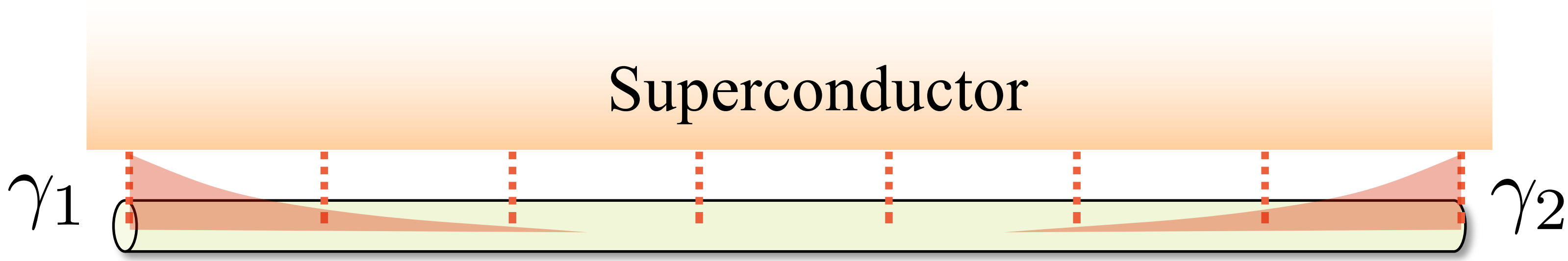
$$H = \sum_j -(2t+2\Delta)i\alpha_j\beta_{j+1} + (2t-2\Delta)i\beta_j\alpha_{j+1} - \mu(i\alpha_j\beta_j - i\beta_j\alpha_j).$$

Majoranas at the ends $(t = \Delta)$:

$$\gamma_L = \beta_1, \gamma_R = \alpha_N$$

$$\gamma_{R,L}^\dagger = \gamma_{R,L}.$$

Majorana zero modes in a topological superconductor



- **Gapped system, two degenerate ground states,** **characterized by having a different** **fermion parity**
- **Defects** **(in this case, the edges of the system) carry protected** **zero modes**
- **Ground state degeneracy is** **"topological"****: no local measurement can distinguish between the two states!**
- **Useful as a "quantum bit"?**

Kitaev (2001), Oreg (2009), Lutchyn (2009),...

Experimental realizations and signatures

Rule of thumb: whenever we have a single Fermi surface in the normal state, if we manage to gap it, we will get a topological superconducting state.

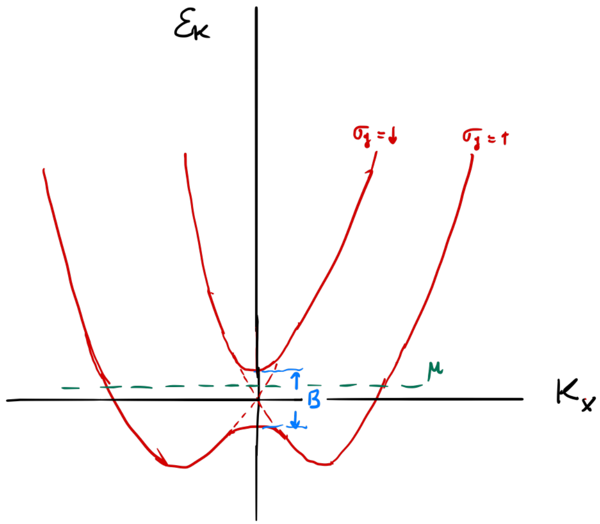
Rule of thumb (2): the topological and trivial states must be separated by a gap closing either at $k = 0$ or $k = \pi$.

Quantum wire with spin-orbit coupling proximity coupled to a superconductor (Oreg et al., Lutchyn et al. (2010)):

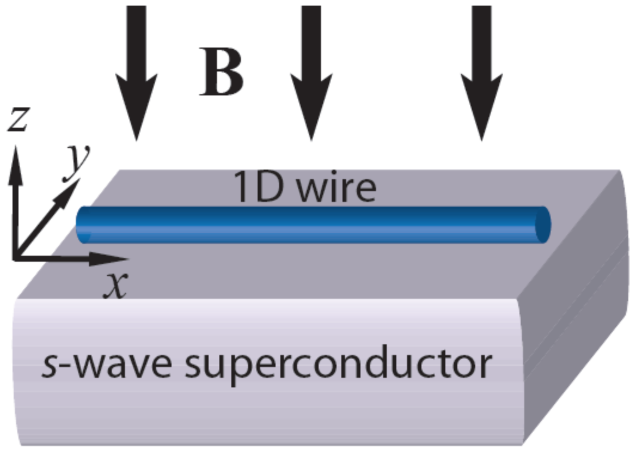
$$H = c^{\dagger}_{k_x} \left(\frac{k_x^2}{2m} - \alpha k_x \sigma^y - \mu - B\sigma^x \right) c_{k_x} + \Delta c^{\dagger}_{k_x\uparrow} c^{\dagger}_{k_x\downarrow}$$

Topological transition at:

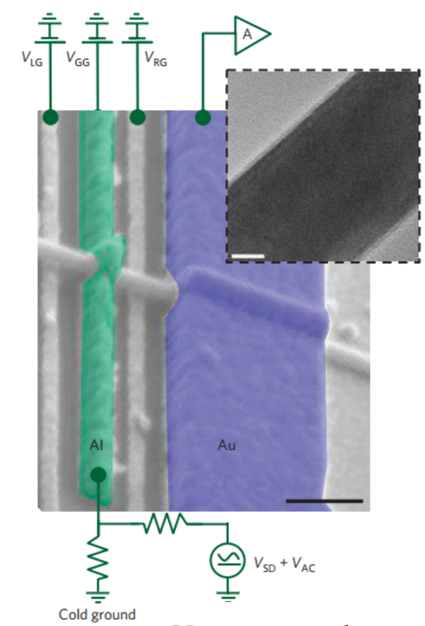
$$B = \pm\sqrt{|\Delta|^2 + \mu^2}$$

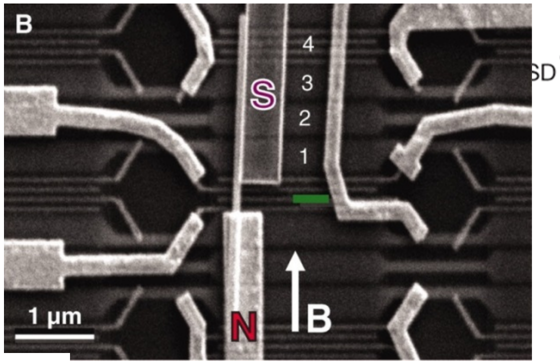


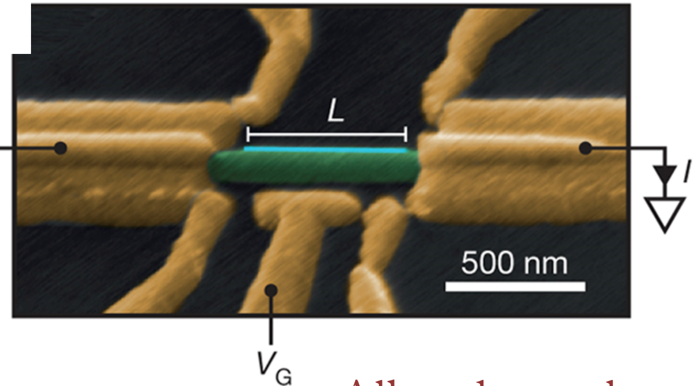
Experimental realizations and signatures



Lutchyn et al. PRL 2010
Oreg et al. PRL 2010







Albrecht et al., Nature 2016

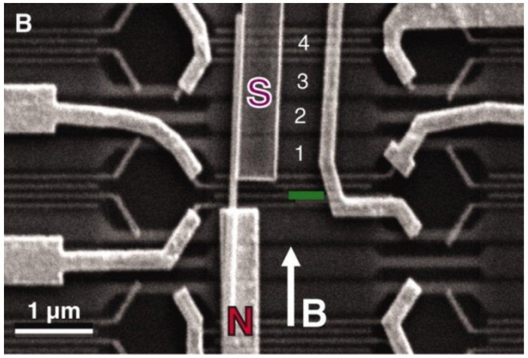
Das et al., Nature Physics 2012

Mourik et al., Science 2012

Rokhinson et al., Nature Phys. (2012), Deng et al., Nano Lett. (2012), Churchill et al., Phys. Rev. B (2013), Nadj-Perge, Science (2014)

Experimental signatures

Zero-bias peak in conductance $G(V) = \frac{dI}{dV}$ from normal metal.



Ideally, $G(V \to 0) = \frac{2e^2}{h}$ if there is only one channel in the metal coupled to the superconducting wire. Zhang, Kouwenhoven et al. (2018)

4π periodic Josephson effect between two topological SC $H_J = i\Gamma e^{i\phi/2}\gamma_1\gamma_2 + H.c. = i\Gamma\gamma_1\gamma_2\cos(\phi/2).$

Wiedenmann, Molenkamp et al. (2016)

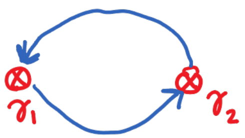
Disappearance of even-odd effect in electron addition spectrum:

$$E(N, V_G) = \frac{e^2(N - CV_G)^2}{2C} + f(N).$$

Usually in a superconductor, $f(N) = \Delta\frac{1-(-1)^N}{2}$; in a topological SC, $\Delta = 0$.

Albrecht, Marcus et al. (2016)

Braiding Majorana zero modes



How do the operators $\gamma_{1,2}$ transform under braiding?

$$\gamma_1 \to \gamma_1' = U_{12}^\dagger \gamma_1 U_{12}$$

$$\gamma_2 \to \gamma_2' = U_{12}^\dagger \gamma_2 U_{12}$$

U_{12} : Unitary adiabatic evolution operator.

We expect

$$\gamma_1' \propto \gamma_2 \quad \text{(up to a phase)}$$

$$\gamma_2' \propto \gamma_1$$

$(\gamma_1')^2 = U_{12}^\dagger \gamma_1 U_{12} U_{12}^\dagger \gamma_1 U_{12} = 1 \Rightarrow$ phases are ± 1

suppose $\gamma_1' = \gamma_2$

transformation has to conserve $i\gamma_1\gamma_2$ (fermion parity of 1,2)

$\to \quad \gamma_2' = -\gamma_1$!

One can check that the transformation that does this is

$$U_{12} = e^{i\phi} \, e^{\frac{\pi}{4}\gamma_1\gamma_2}$$

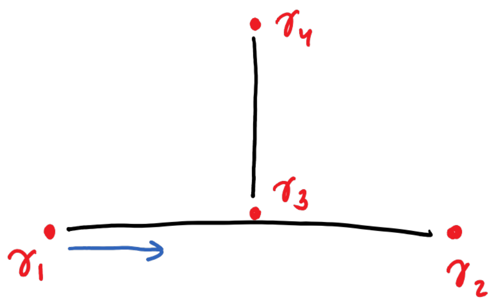
$e^{i\phi}$: phase we can't determine from the present considerations.

Braiding Majorana zero modes (2)

(Alicea, Oreg, von Oppen, Refael, Fisher 10')

"exchange" γ_1, γ_2 ?

T - junction geometry: (Two "auxiliary" Majoranas γ_3, γ_4:)



Change position of γ_i (e.g., by applying gate potentials):

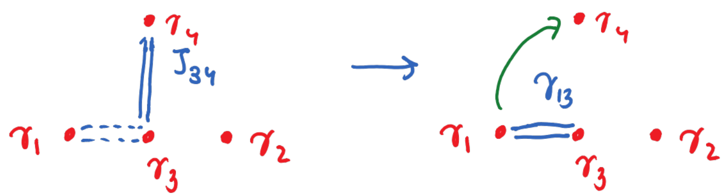
Braiding Majorana zero modes (3)

$$H_{eff} = \sum_{i,j} i\, J_{ij}(t)\, \gamma_i \gamma_j$$

$$i\gamma_1\gamma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^x$$

$$i\gamma_3\gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma^y$$

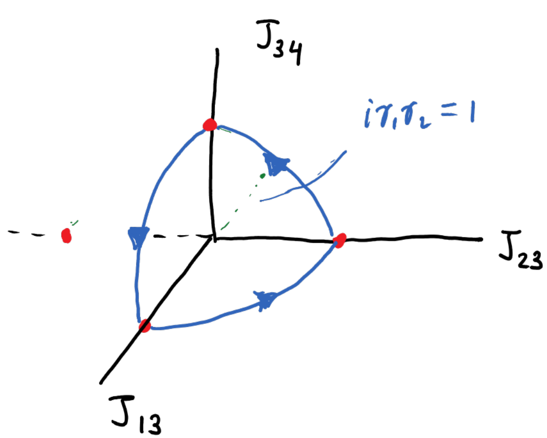
$$i\gamma_3\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^z$$

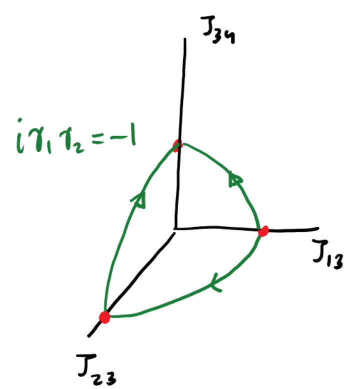


The braiding protocol can be visualized as follows.

The Hamiltonian has 3 parameters, J_{13}, J_{23}, and J_{34}.

Draw them in a 3D space and fix $J_{13}^2 + J_{23}^2 + J_{34}^2 = J^2$.





Berry phase $= \pm \frac{1}{2} \cdot \frac{1}{8}\, 4\pi = \pm \frac{\pi}{4}$

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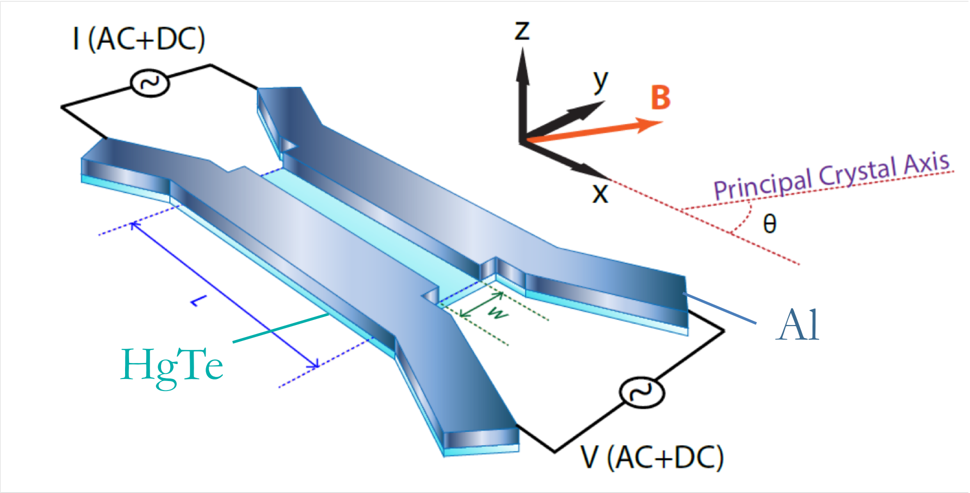
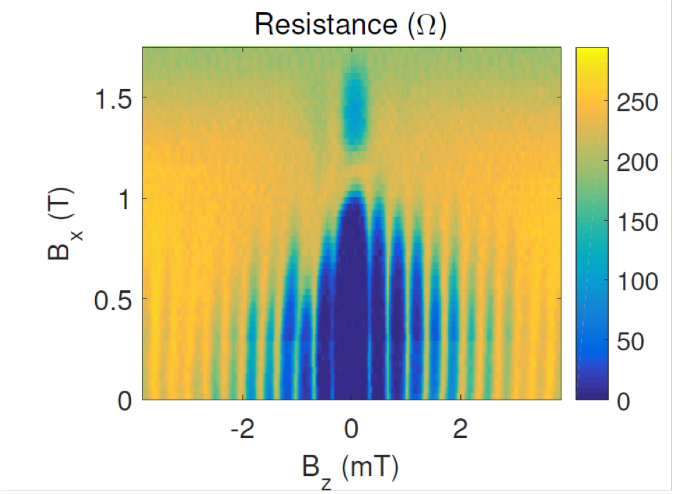
- New platform: topological superconductivity in planar Josephson junctions
- Beyond Majoranas

New platform: planar Josephson junctions

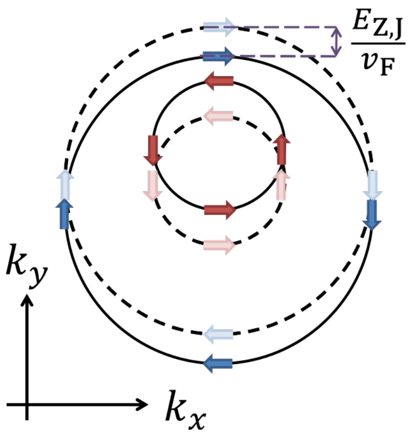
Hart et al. experiment (Yacoby group, 2017):



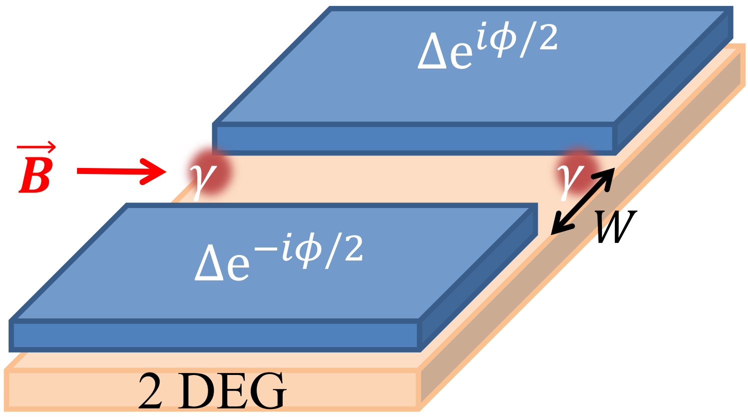
As a function of B_x, the critical current vanishes and recovers:



Shifted Fermi surfaces due to Zeeman field $B \parallel x$:



New platform: planar Josephson junctions



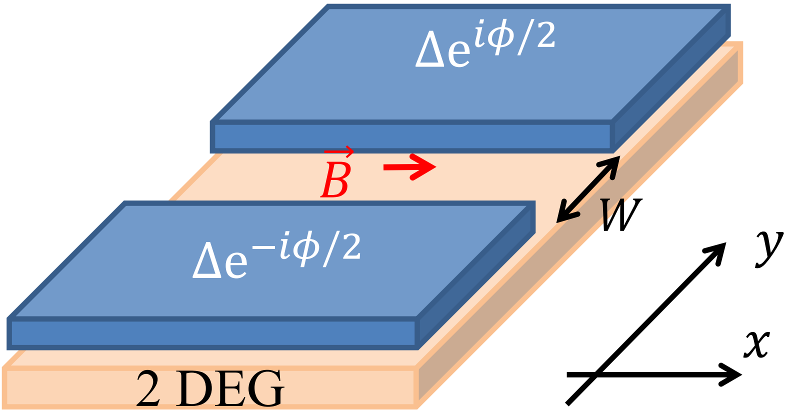
Ingredients:
✓ 1D
✓ Spin-orbit
✓ Superconductivity
✓ Magnetic field

New features:

- Robust topological phase, weak dependence on chemical potential
- Can tune itself the topological phase!

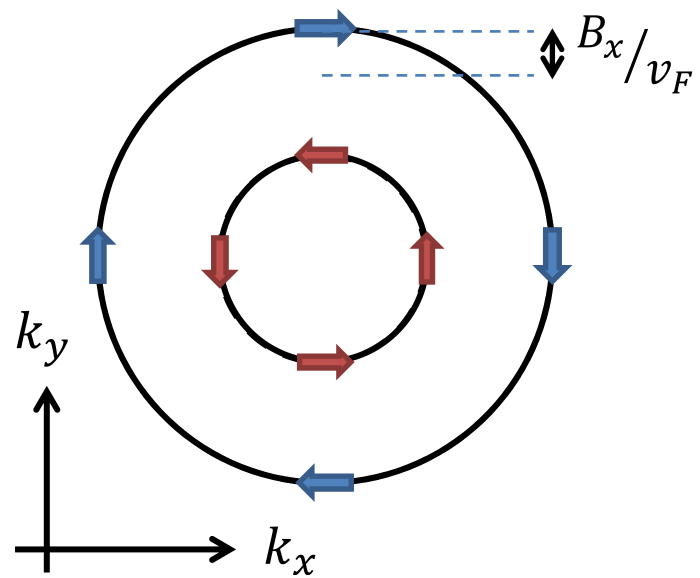
Pientka, Keselman, EB, Yacoby, Stern, Halperin (PRX, 2017); Hell, Leijnse, Flensberg (PRL, 2017)

Setup and Model



Hamiltonian in the normal region:

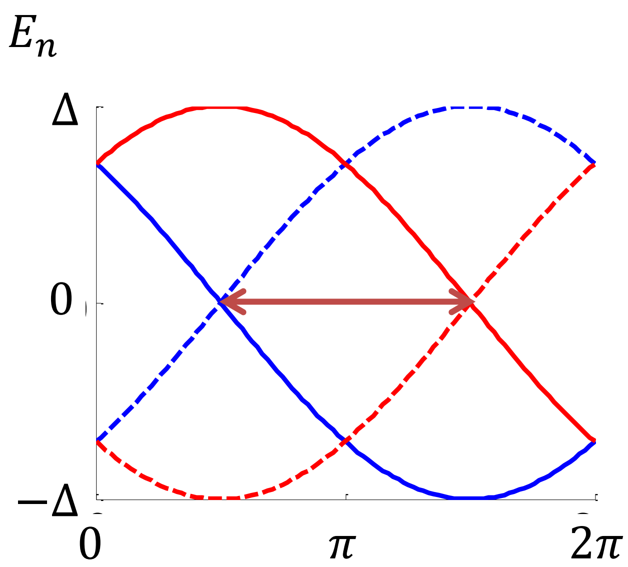
$$H_0 = \frac{k_x^2 - \partial_y^2}{2m} - \mu$$
$$+\alpha\left(k_x\sigma_y + i\partial_y\sigma_x\right) + B_x\sigma_x$$



Phase Diagram

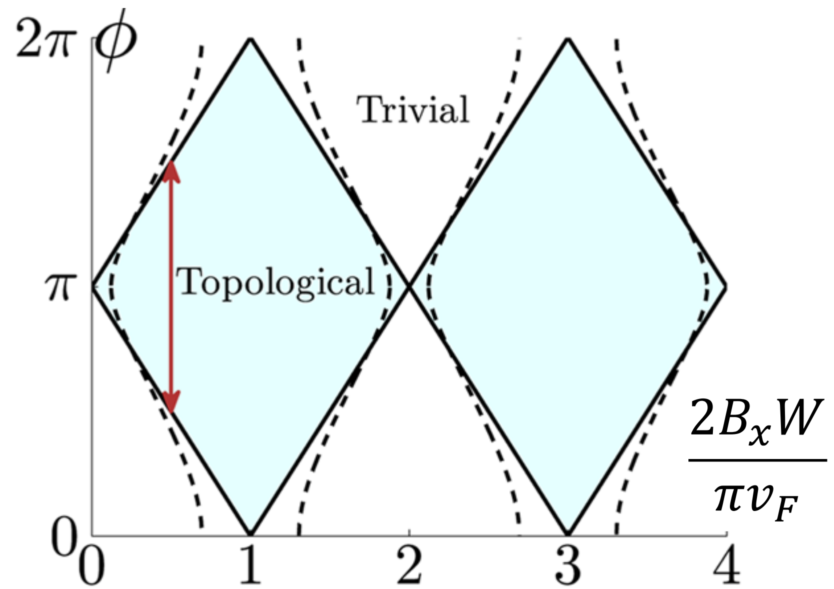
k_x=0 bound states:

$$E_n = \Delta\cos\left(\frac{\phi}{2} \pm \frac{B_x}{v_F} W\right)$$



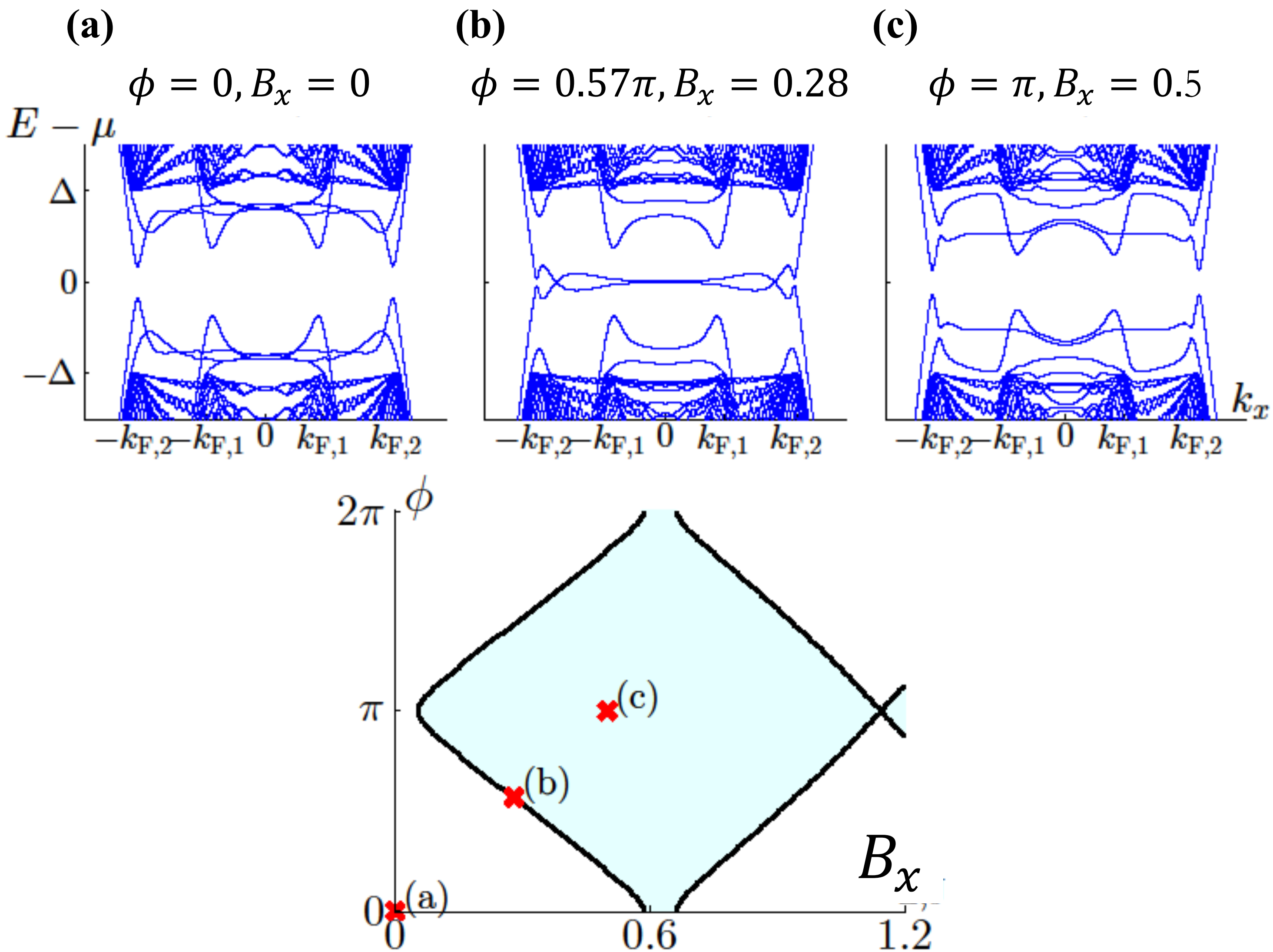
Gap closing lines (for any W):

$$\phi \pm 2\frac{B_x}{v_F} W = (2n+1)\pi$$



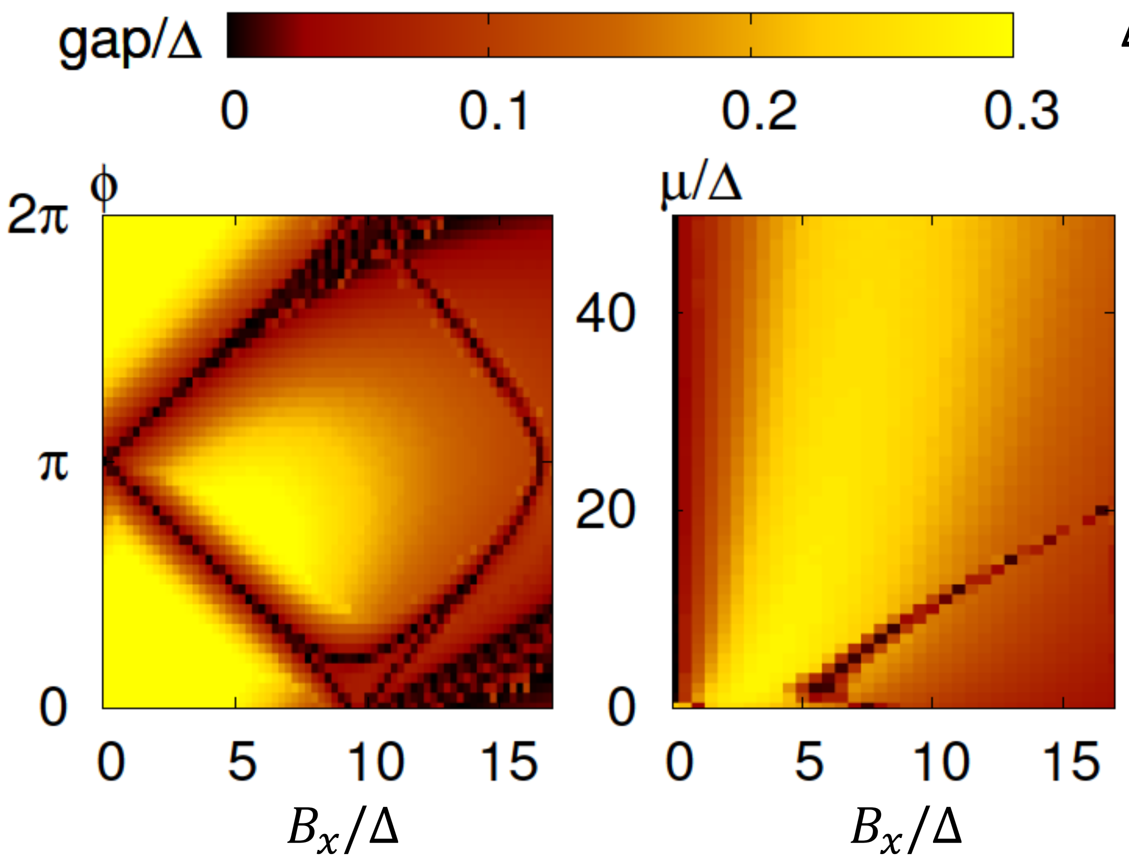
No explicit dependence on μ!

Spectrum across the phase transition



Gap in the system

$\Delta \simeq 1/_{mW^2}$



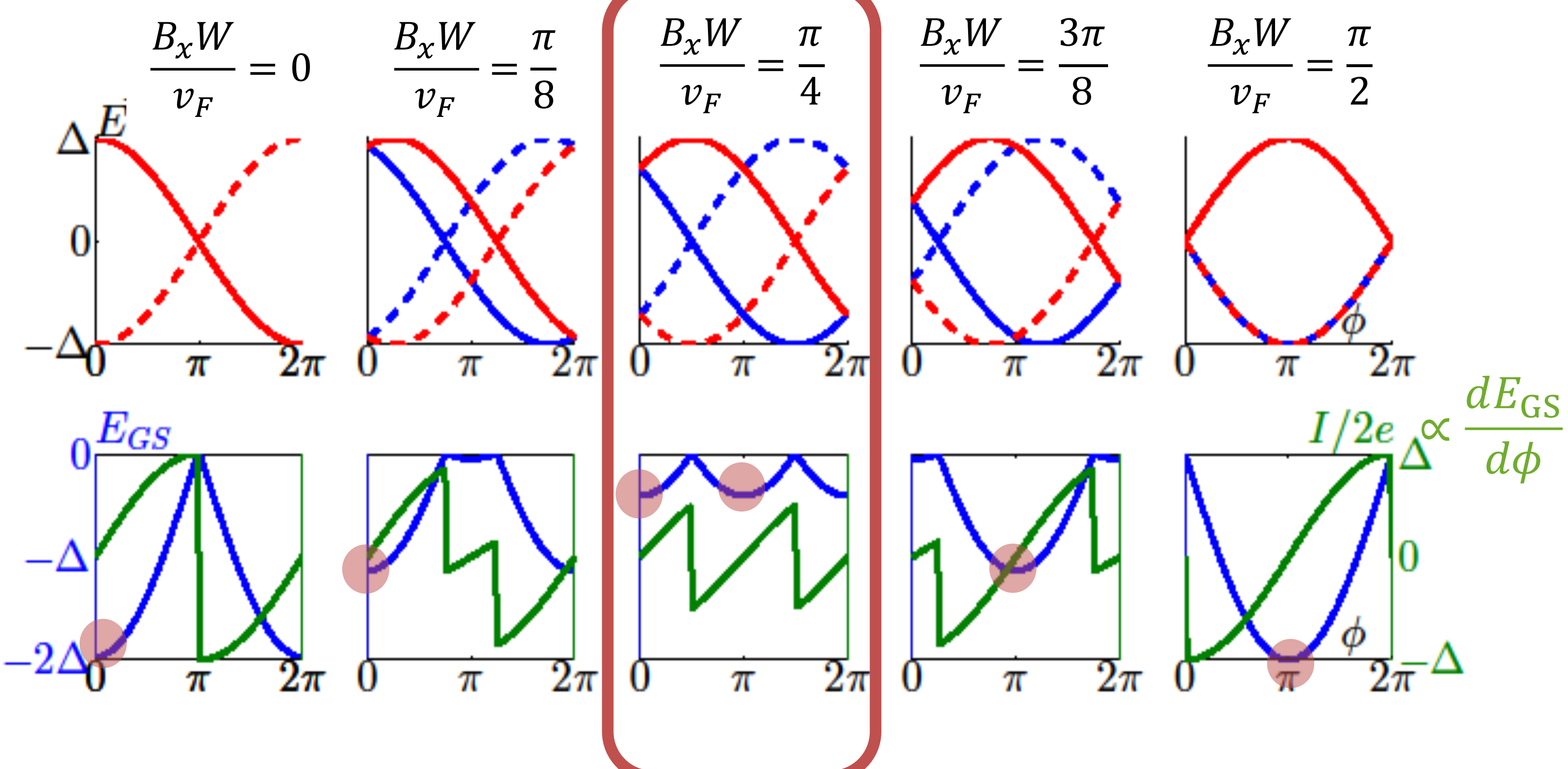
1st-order topological phase transition

Consider a system with no phase bias. What happens as B_x is varied?

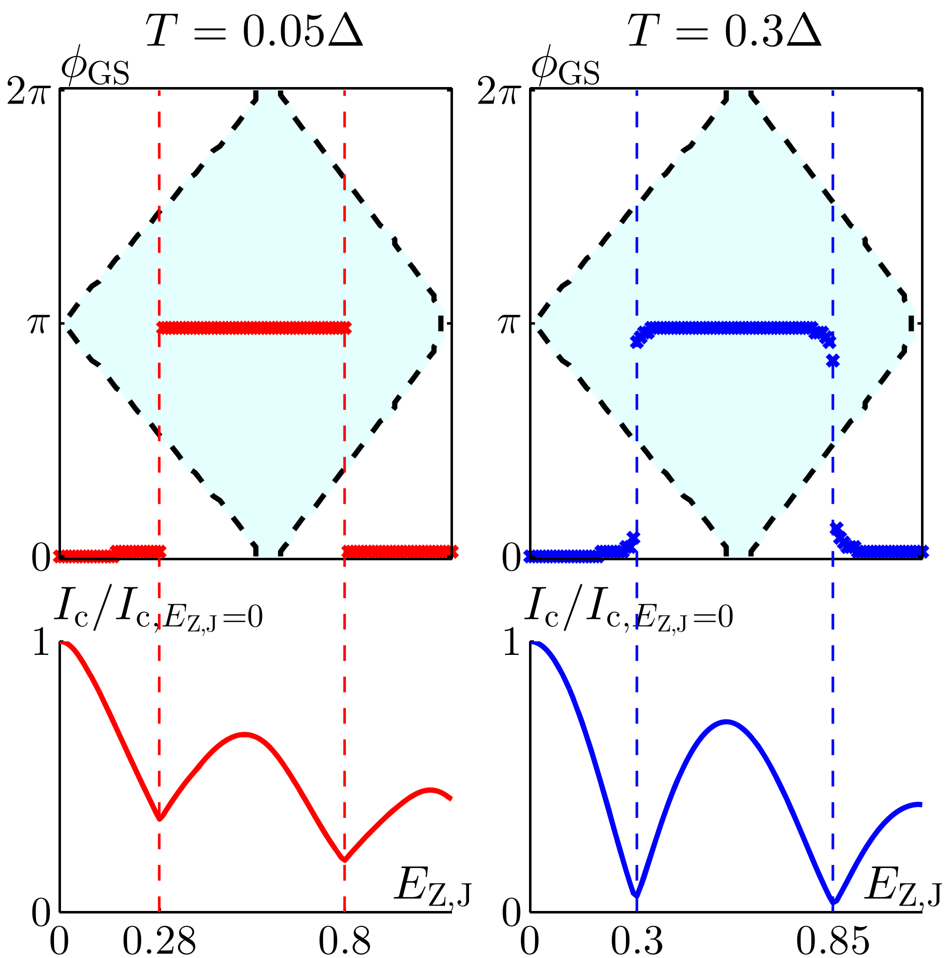
1st-order topological phase transition

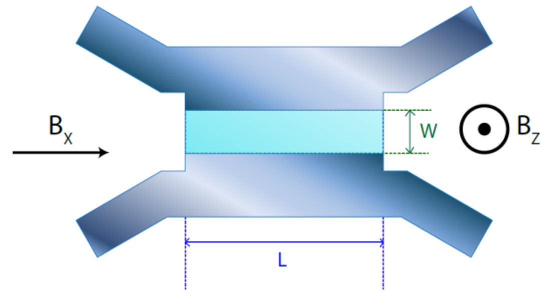
Consider a system with no phase bias. What happens as B_x is varied?

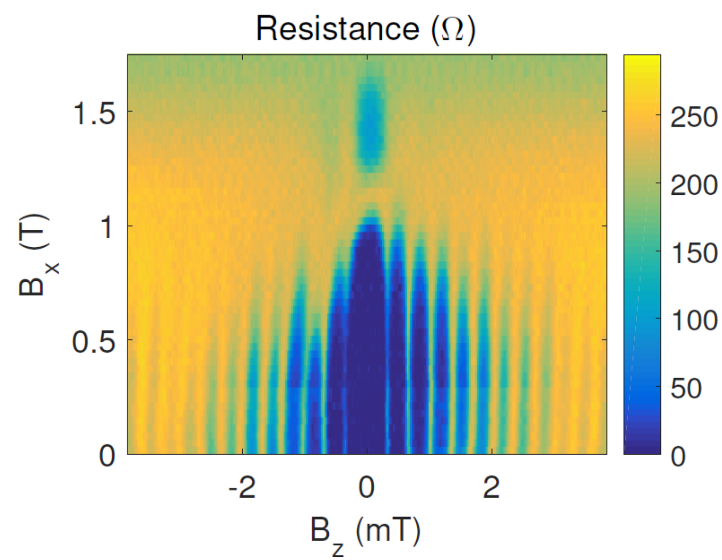
$k_x = 0$ mode bound states $\quad E_n = \Delta\cos\left(\frac{\phi}{2} \pm \frac{B_x}{v_F} W\right)$



Critical current and 1st order transition

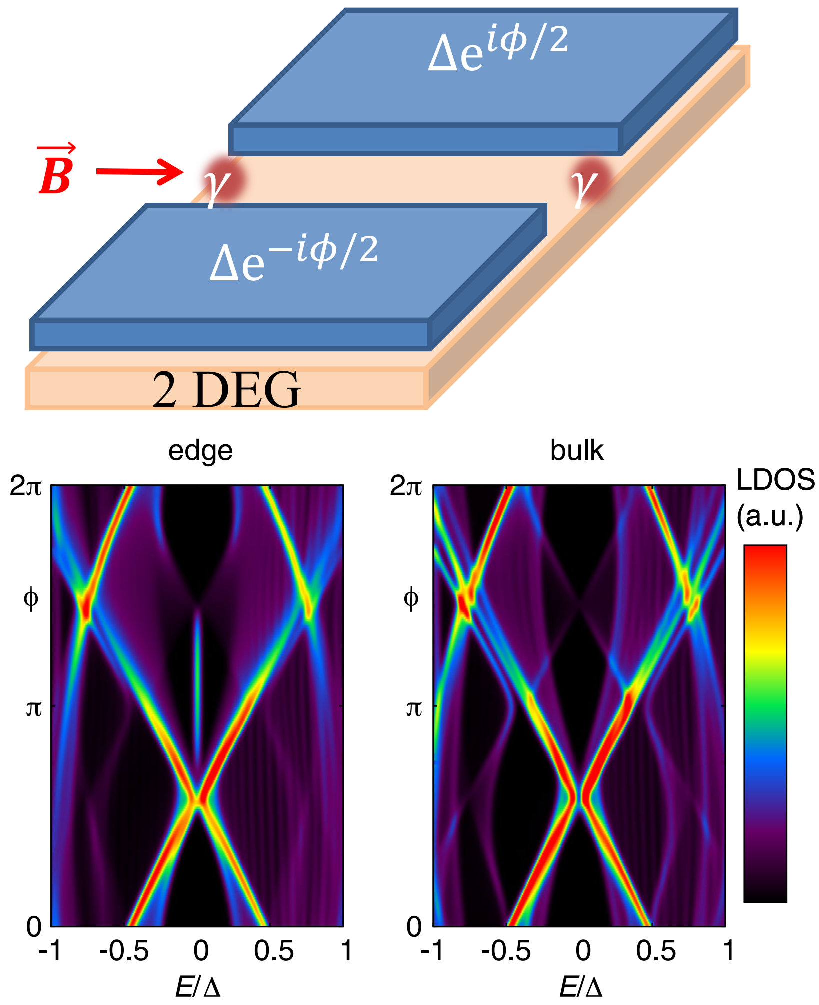






Hart et al. Nature Phys. (2017)

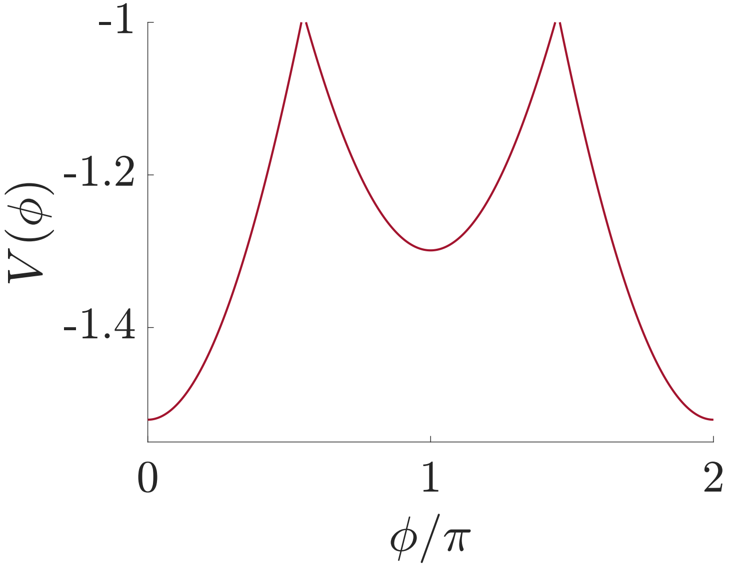
Majorana zero modes

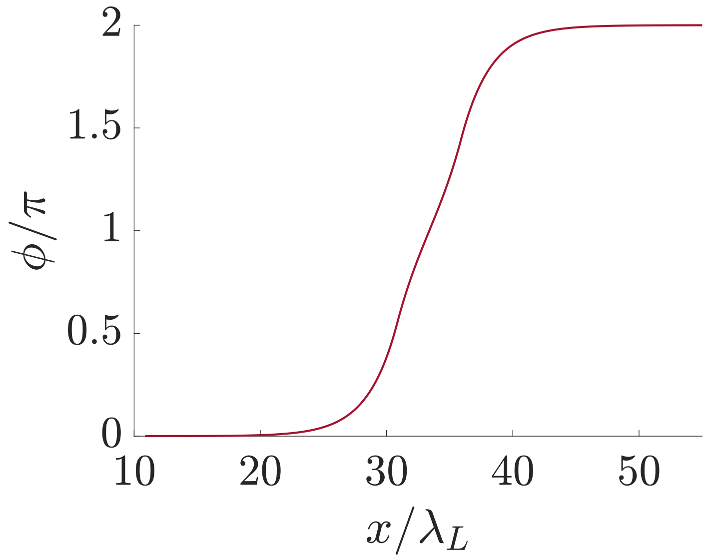


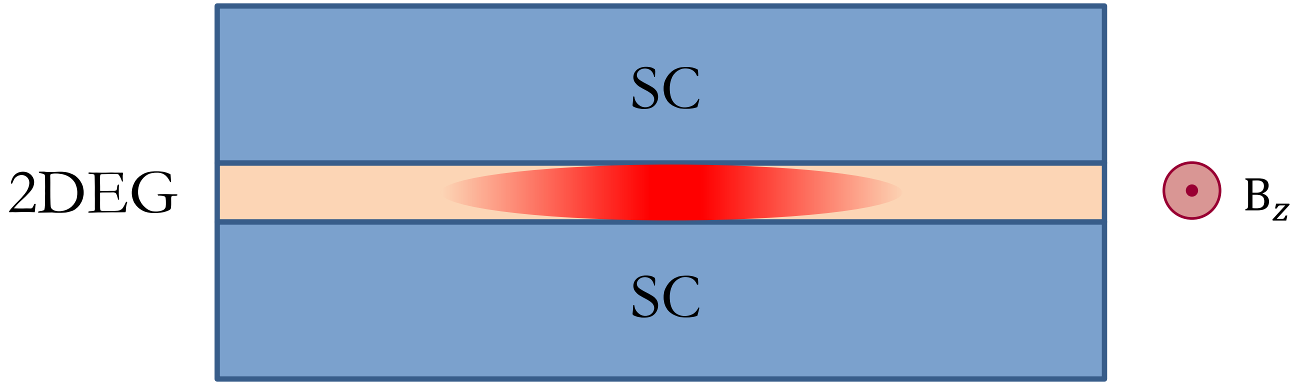
Zero bias peak at the edge and phase-dependent gap closings

A. Fornieri et al. (Nature, 2019), H. Ren et al. (Nature, 2019)

Fractional Josephson vortices

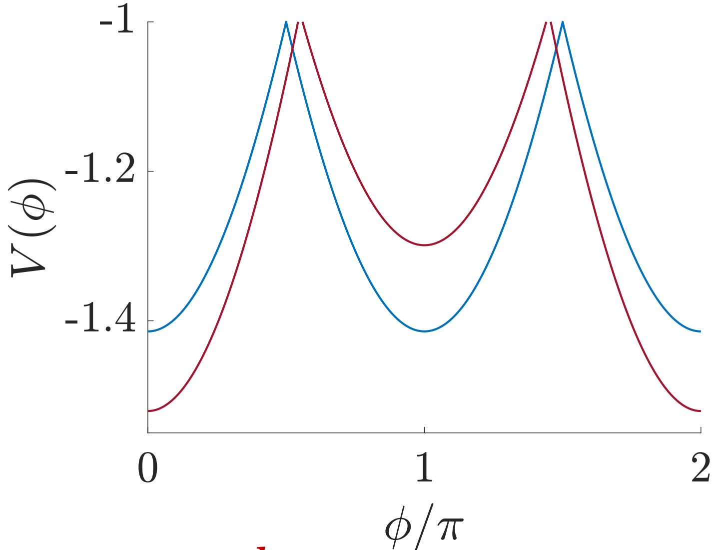


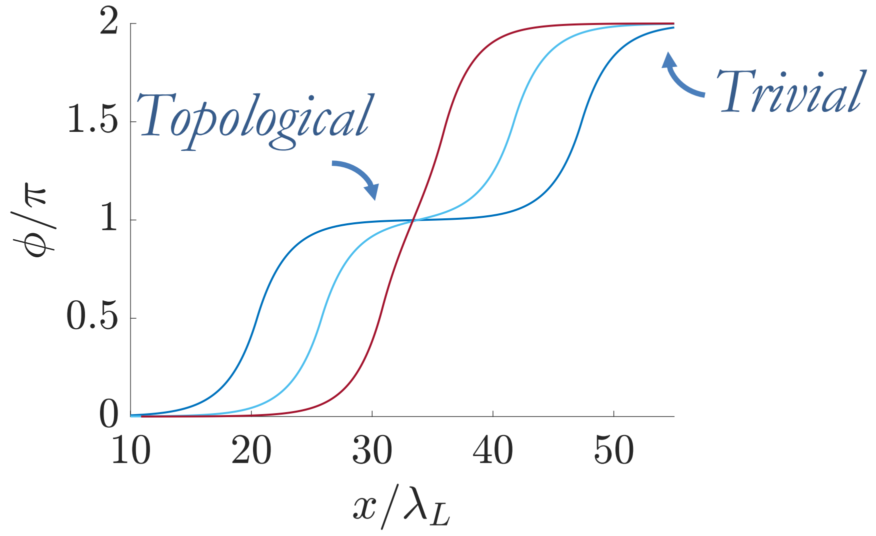




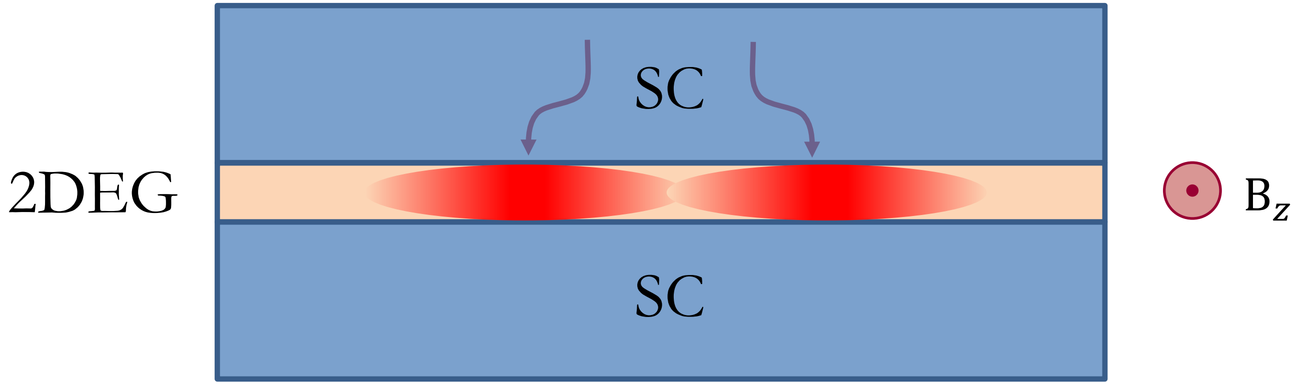
A. Stern, EB (PRL, 2019)

Fractional Josephson vortices



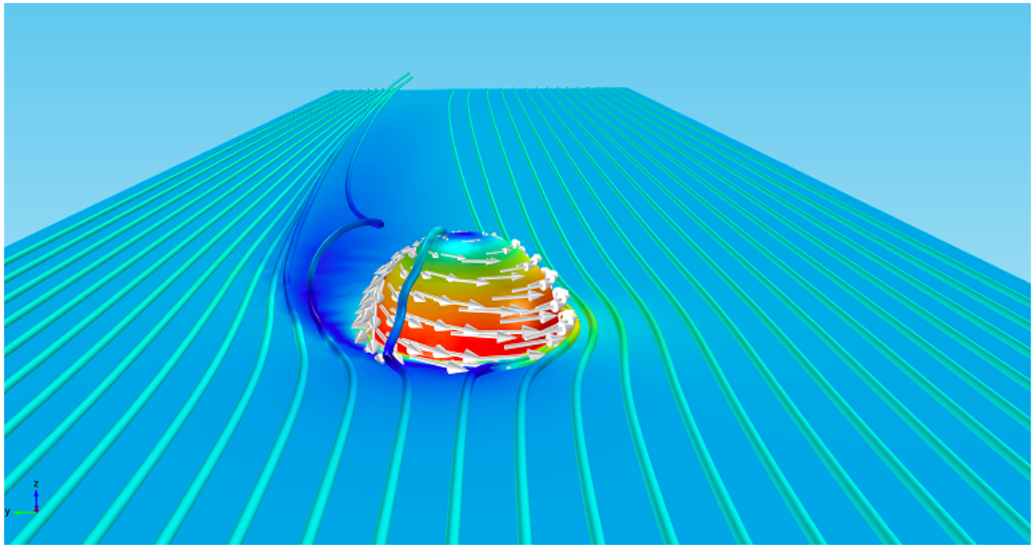


$\sim \frac{h}{4e}$ vortices carrying Majorana zero modes!



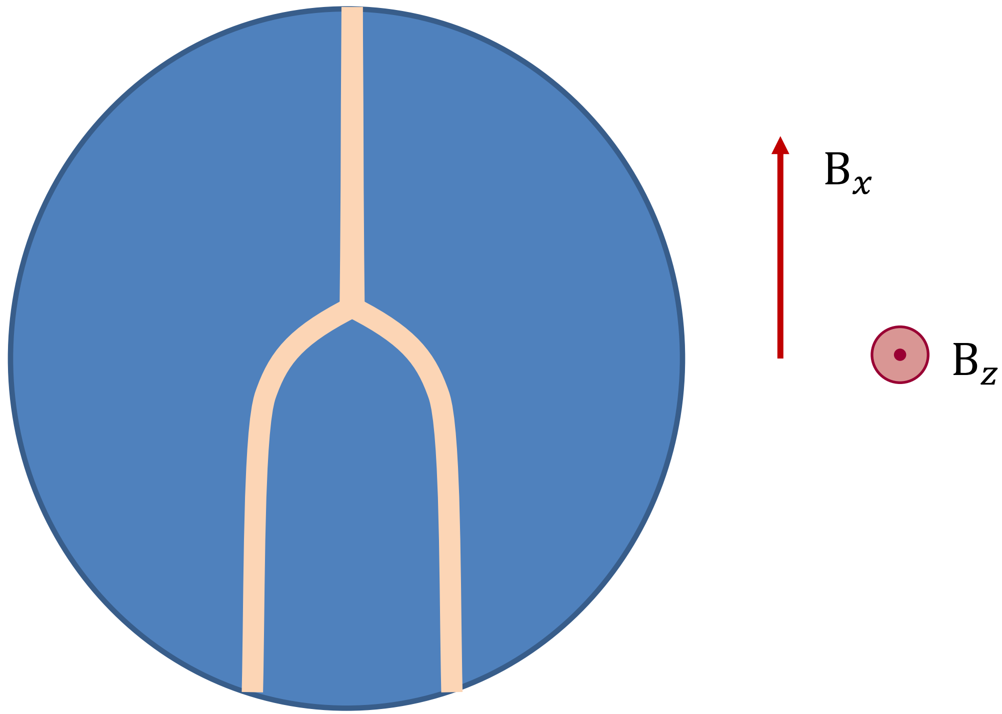
A. Stern, EB (PRL, 2019)

Controlling Majoranas by supercurrents?



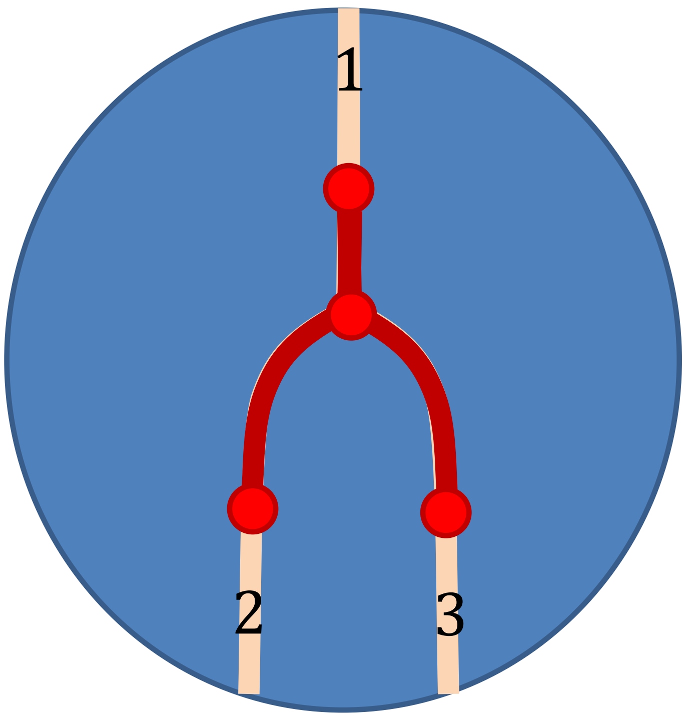
A. Stern, EB (PRL, 2019)

Tri-junction geometry



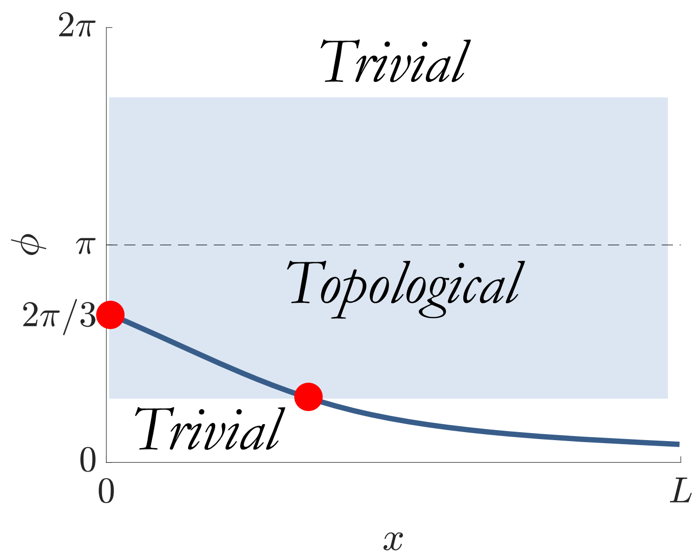
A. Stern, EB (PRL, 2019)

Tri-junction geometry



$\phi_{i=1,2,3}(x)$: Gauge-invariant phases across junctions

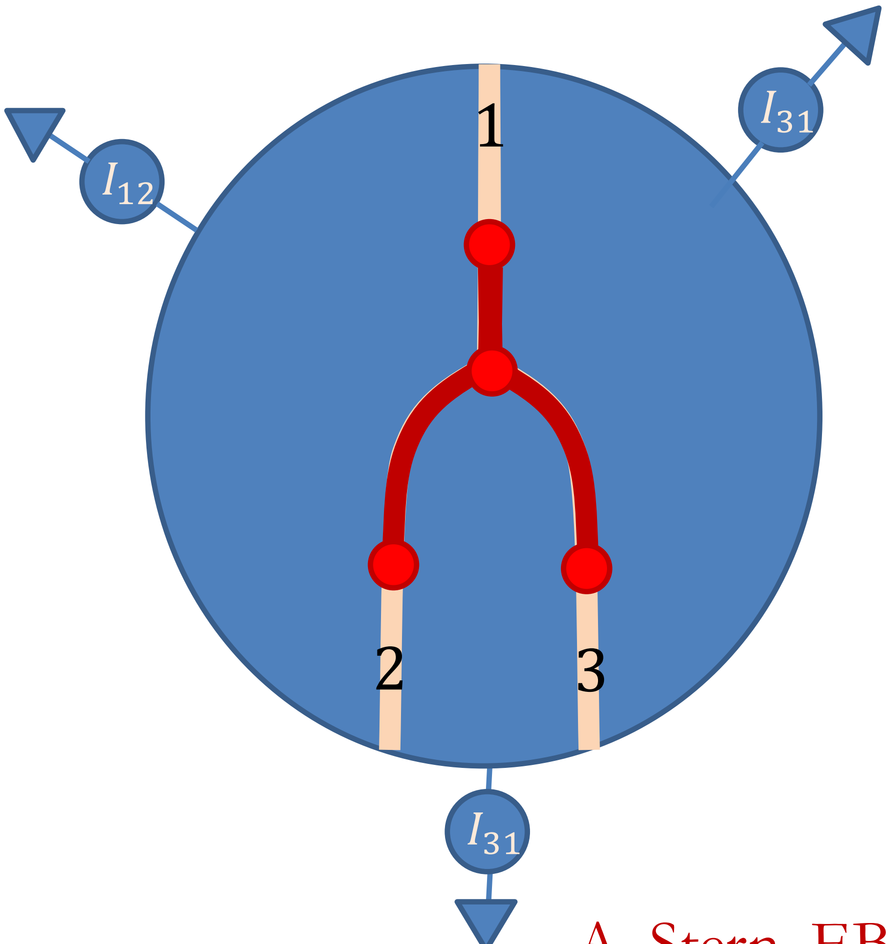
$$\phi_1 + \phi_2 + \phi_3 = 2\pi n$$

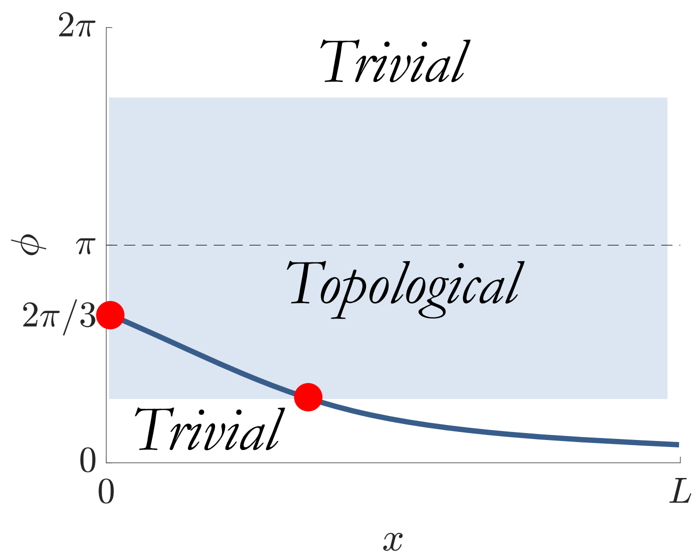


A. Stern, EB (PRL, 2019)

Manipulations by supercurrents

$$I_{12} + I_{23} + I_{31} = 0$$





A. Stern, EB (PRL, 2019)

Manipulations by supercurrents

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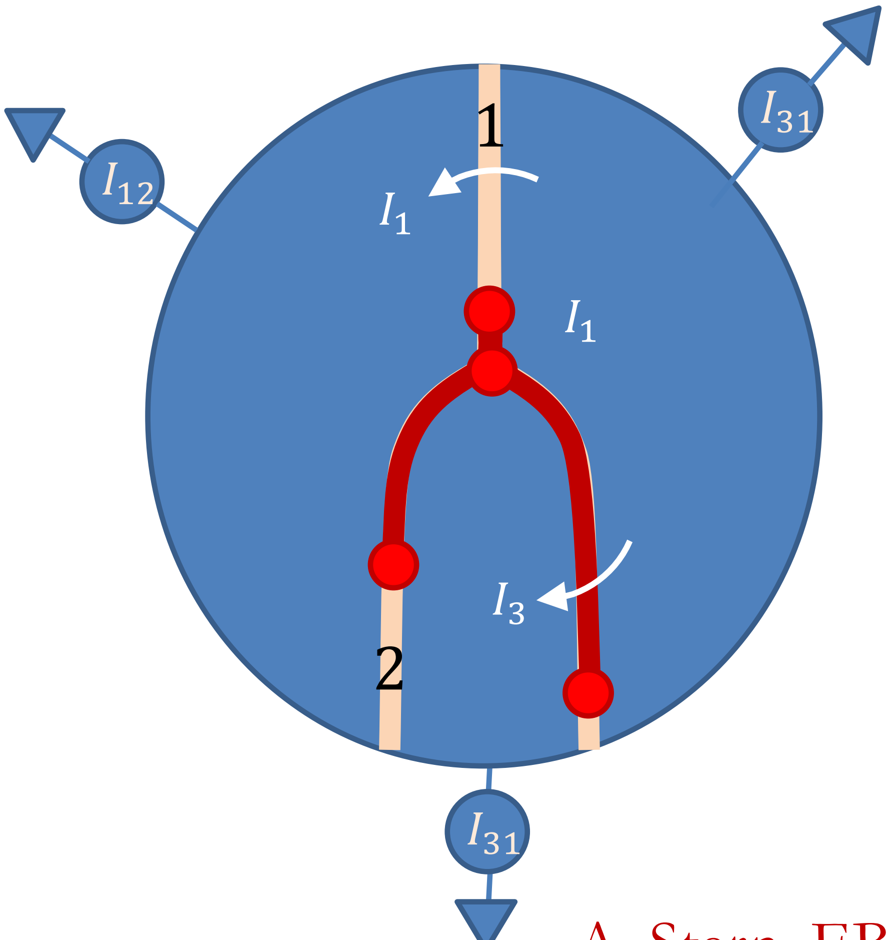
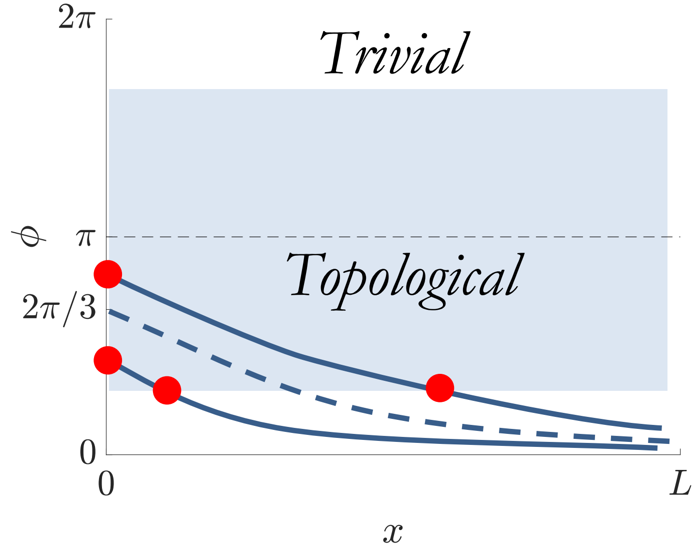


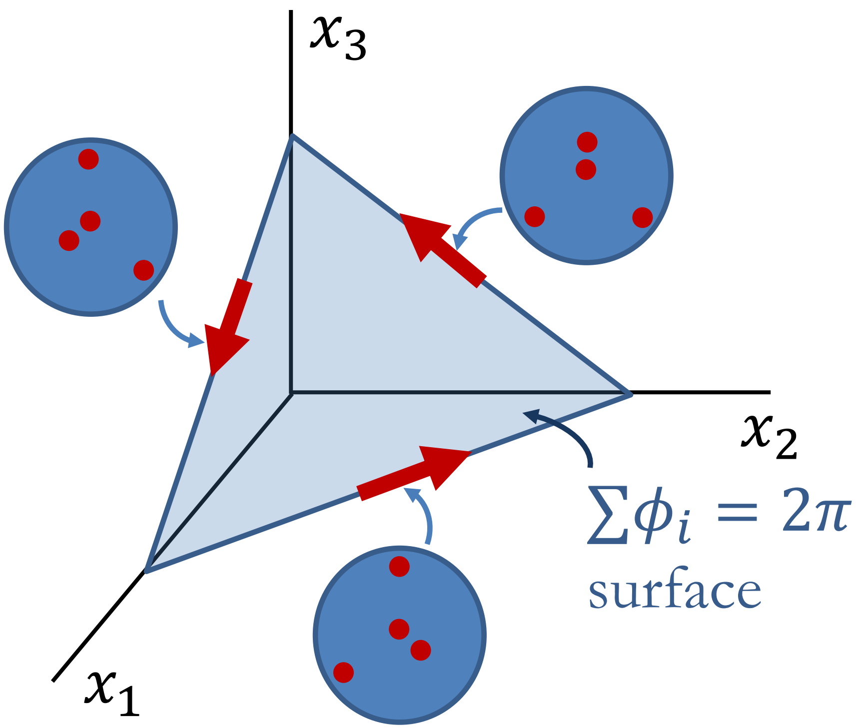


A. Stern, EB (PRL, 2019)

Control by supercurrents

$x_{i=1,2,3}$: distance of ith Majorana from center

To braid, we need to encircle a line in a three-dimensional parameter space.



The red cycle implements braiding of Majoranas.*

*Assuming the coupling between each pair is monotonic in the separation.

A. Stern, EB (PRL, 2019)

Beyond Majoranas

How can one get protected zero modes with richer non-Abelian properties?

Not in 1D systems – not even with interactions.

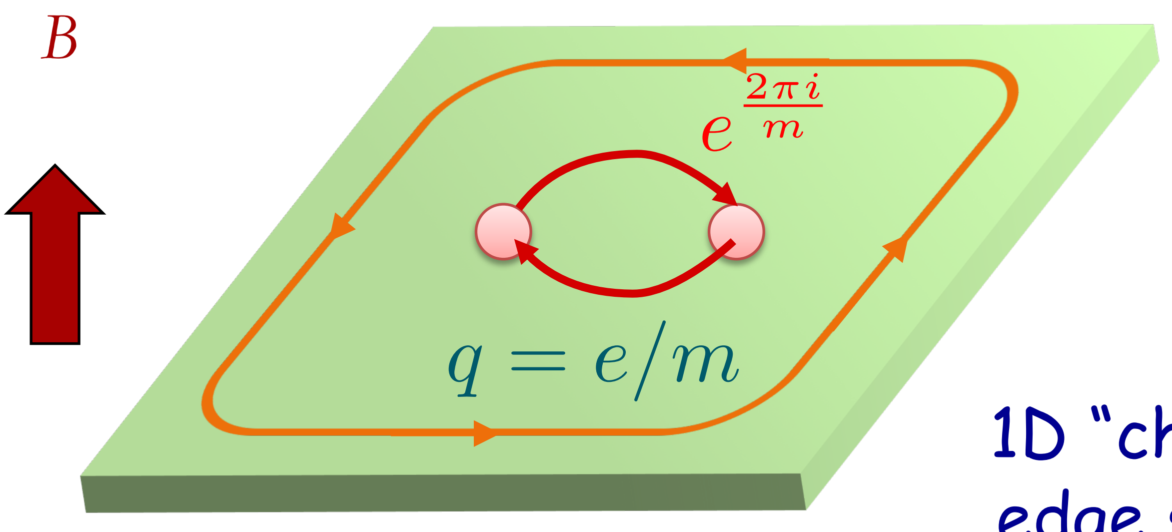
Non-Abelian Fractional quantum Hall states:
Moore-Read ($\nu = \frac{5}{2}$), Read-Rezayi,...

Alternatively: start from the edge states of Abelian FQH states to engineer new zero modes

Beyond Majoranas

Consider the effectively 1D boundaries of 2D a topological phase which supports (abelian) anyons.

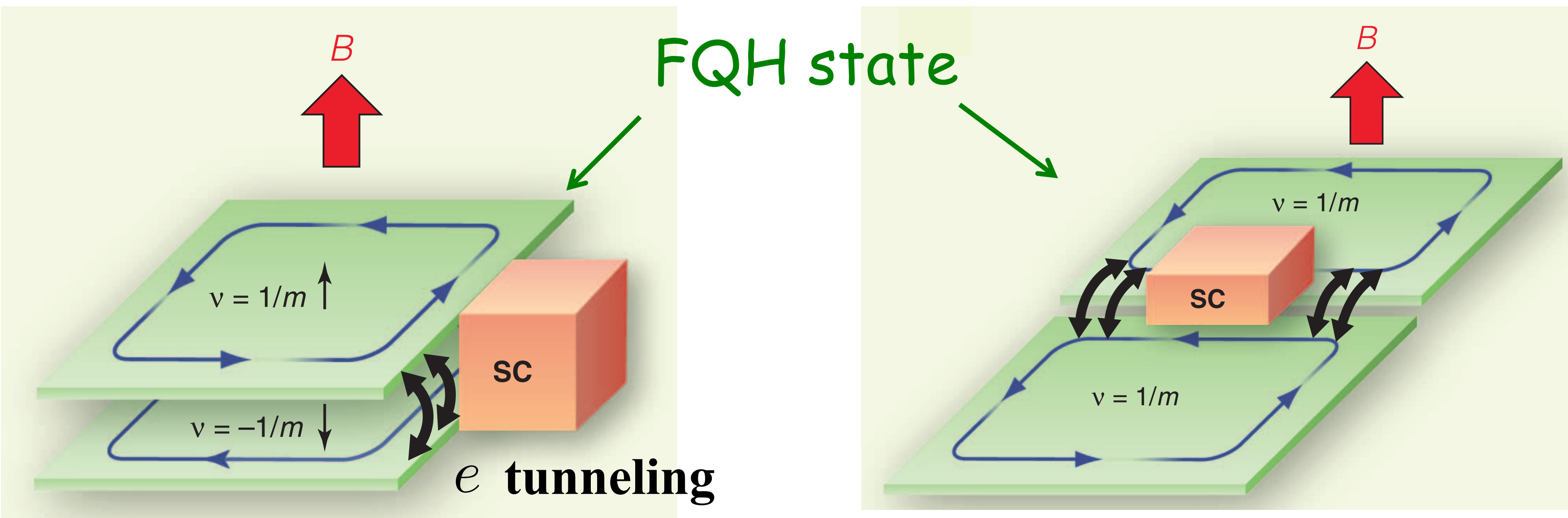
For example:
ν = 1/m Fractional Quantum Hall (Laughlin) state



1D "chiral" edge state

Beyond Majorana zero modes

Setups for fractionalized Majorana zero modes:



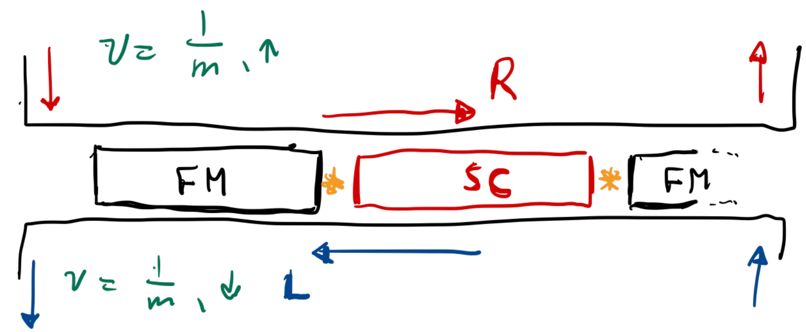
Lindner, EB, Stern, Refael (PRX, 2012);
Clarke, Alicea, Shtengel (Nature Comm., 2013);
Cheng (PRB, 2013)

Fractionalized Majorana (parafermion) modes

Effective edge theory:

$$H = \frac{m}{2\pi} u \int dx \left[K (\nabla\phi)^2 + \frac{1}{K} (\nabla\theta)^2 \right]$$

$$- \int dx \, g_S(x) \underbrace{\cos(2m\phi)}_{\psi_R \psi_L + h.c.} - \int dx \, g_B \underbrace{\cos(2m\theta)}_{\psi_R^\dagger \psi_L + h.c.}$$



quasiparticle operator: $\chi_{R,L} \sim e^{i(\phi \pm \theta)}$

Electron operator: $\psi_{R,L} \sim e^{im(\phi \pm \theta)}$

Commutation relation: $[\phi(x), \theta(x')] = i\frac{\pi}{m}\Theta(x'-x)$

Charge density: $\rho = \frac{1}{\pi}\partial_x\theta$

Three distinct phases of the edge:

1. Gapless
2. Gapped, g_B dominated: $\langle e^{2i\theta}\rangle \neq 0$
3. Gapped, g_S dominated: $\langle e^{2i\phi}\rangle \neq 0$

Between 2 and 3, a new type of "fractionalized Marjoana" zero mode!

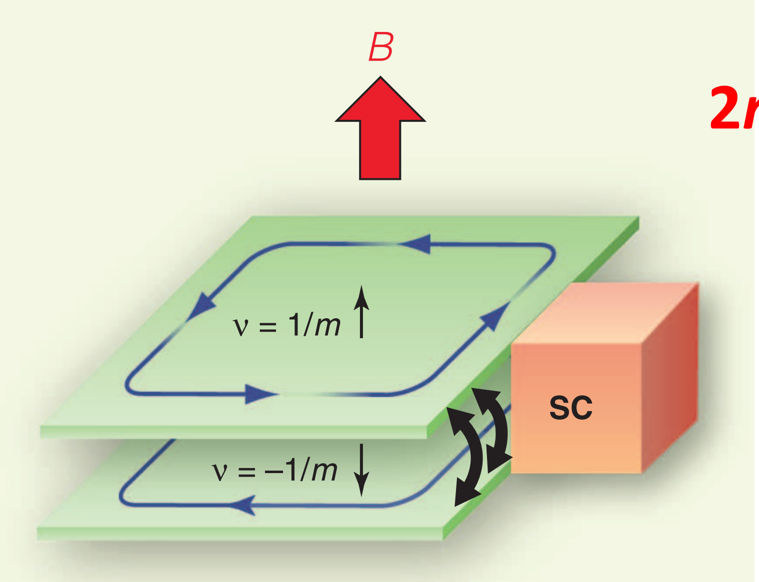
(At these points, a Laughlin q.p. can be injected at no energy cost)

Ground state degeneracy

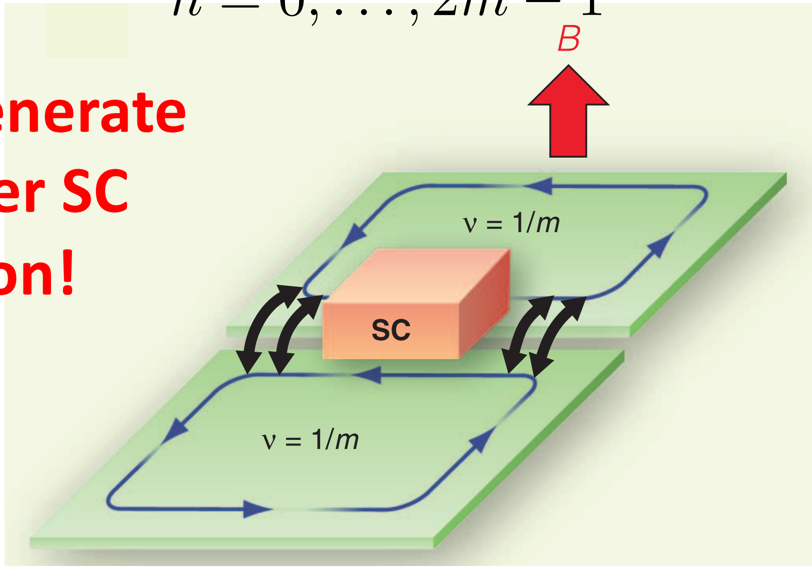
Fermion parity conservation in SC region: $$[e^{i\pi\hat{Q}}, H] = 0$$

On the edge of a fractional quantum Hall phase, Q is fractional: $$\hat{Q} = \frac{n}{m}$$

$$n = 0, \ldots, 2m-1$$



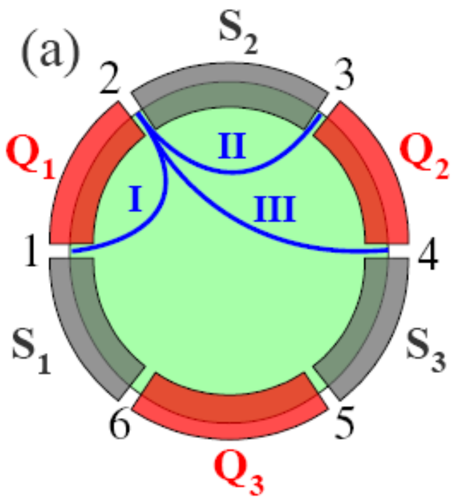
2*m* degenerate G.S. per SC region!



Lindner, EB, Stern, Refael (PRX, 2012); Clarke, Alicea, Shtengel (Nature Comm., 2013); Cheng (PRB, 2013)

Braiding

Braiding domain walls 3 and 4:



$$U_{34} = \exp\left(i\frac{\pi m}{2}\hat{Q}_2^2\right) = \exp\left(i\frac{\pi}{2m}q_2^2\right)$$

$$Q_2 = \frac{1}{m}q_2, \qquad q_2 = 0, \ldots, 2m-1$$

Example: m=3 $\quad q_2 = 2p + 3q \quad (p = 0, 1, 2, \;\; q = 0, 1)$

$$U_{34} = \exp\left(i\frac{\pi}{6}q_2^2\right) = \exp\left(-i\frac{\pi}{2}q^2\right)\exp\left(i\frac{2\pi}{3}p^2\right)$$

(Majorana) ⊗ (Something new!)