$H = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$ $\alpha \neq 1$

Topology in non-Hermitian systems



Emil J. Bergholtz

Knut och Alice Wallenbergs Stiftelse

Vetenskapsrådet



Credit to Flore Kunst, Elisabet Edvardsson, Marcus Stålhammar, Johan Carlström and Jan Budich

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Non-Hermitian, really?

 Complex energies, non-unitary time-evolution, ..., Pandora's box!?

Relevance:

- Dissipative systems experiments!
- Various classical mechanical, electrical, robotic and optical metamaterials
- Photonic systems with gain and/or loss
- Open, non-equilibrium systems toy alternative to the Lindblad master equation
- Effective description of systems with finite lifetime states

Need:

Basic theory!





 $\operatorname{Im}[E] \sim 1/\tau$

Goals of today

- Provide a pedagogical account of non-Hermitian systems as seen from the <u>perspective</u> of condensed matter theory and topological phases
 - Simple but profound non-Hermitian aspects
 - Point to recent <u>experiments</u>
 - Figures and references...

• Convince you that it is an interesting topic!

Plan:

Minimal examples:

1:
$$H = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$$
 $\alpha \neq 1$ 2: $H = \sum_{i} \left(J_L c_i^{\dagger} c_{i+1} + J_R c_{i+1}^{\dagger} c_i \right)$

Focus:

• Exceptional nodal phases

Anomalous bulk-boundary correspondence



Minimal example 1: a two-level system

$$H = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} \qquad \alpha \neq 1$$

Take home: Exceptional degeneracies & Square roots

A simple but useful example $H = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$

• Eigenvalues generally complex

$$E_{\pm} = \pm \sqrt{\alpha}$$

- Winding of α twice yield a winding of E_\pm only once!



Note the branch point and branch cut

- Riemann surface plotted

- Non-orthogonal eigenvectors
 - Left and right eigenvectors are different
 - ...but can be combined to form
 a "bi-orthogonal" set whenever

 $\alpha \neq 0$

$$\Psi_{R,\pm} = \begin{pmatrix} \pm \sqrt{\alpha} \\ 1 \end{pmatrix}$$

$$\Psi_{L,\pm} = \begin{pmatrix} 1 & \pm\sqrt{\alpha} \end{pmatrix}$$

 $\langle \Psi^L_i | \Psi^R_j
angle = \delta_{ij}$ (unusual normalisation)

An exceptional point $(\alpha = 0)$

$$H = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

- Doubly degenerate eigenvalue
- But only one normalisable eigenvector!
 - The left eigenvector is the "opposite"

 $\Psi_{R,\pm} = \begin{pmatrix} 0\\1 \end{pmatrix}$ $\Psi_{L,\pm} = \begin{pmatrix} 1&0 \end{pmatrix}$

 $E_{+} = 0$

- "Exceptional points" (EPs) with singular behaviour
 - Rare, measure zero in the space of matrices
 - Diverging response $|\partial_{lpha} E(lpha)|
 ightarrow \infty$
- When can we expect EPs to occur and what are their consequences?

Focus 1: Exceptional nodal phases



 Take home: Gapless nodal phases are far more abundant and conceptually rich than in the Hermitian realm

A step back: Band crossings in Hermitian systems

• When can we expect two energy bands to cross at a single point?

C. Herring, Phys. Rev. 52 365 (1937)

$$H(\mathbf{k}) = \begin{pmatrix} d_3(\mathbf{k}) + d_0(\mathbf{k}) & d_1(\mathbf{k}) - id_2(\mathbf{k}) \\ d_1(\mathbf{k}) + id_2(\mathbf{k}) & -d_3(\mathbf{k}) + d_0(\mathbf{k}) \end{pmatrix}$$
$$= \mathbf{d}(\mathbf{k}) \cdot \sigma + d_0(\mathbf{k})$$
$$E(\mathbf{k}) = \pm \sqrt{d_1^2(\mathbf{k}) + d_2^2(\mathbf{k}) + d_3^2(\mathbf{k})} + d_0(\mathbf{k})$$

- 3 equations, hence tuning of 3 parameters needed
- Fine-tuning in 2d
- Stable and generic in 3d!
- Simplest case the Weyl Hamiltonian

 $H = v\mathbf{k} \cdot \sigma$

Energy (eV)



Weyl semimetals — 3d counterparts of Chern insulators

Band touching points generic and protected by Chern numbers





 Implies novel "Fermi arc" surface states

X. Wan, A. M. Turner, A. Vishwanath, and S. Y. Savrasov, Phys. Rev. B 83, 205101 (2011)



What about the non-Hermitian case?

$$\begin{split} H(\mathbf{k}) &= \mathbf{d}(\mathbf{k}) \cdot \sigma \quad \text{now with} \quad \mathbf{d}(\mathbf{k}) = \mathbf{d}_{\mathrm{R}}(\mathbf{k}) + i\mathbf{d}_{\mathrm{I}}(\mathbf{k}) \\ E(\mathbf{k}) &= \pm \sqrt{\mathbf{d}_{\mathrm{R}}(\mathbf{k})^2 - \mathbf{d}_{\mathrm{I}}(\mathbf{k})^2 + 2i\mathbf{d}_{\mathrm{R}}(\mathbf{k}) \cdot \mathbf{d}_{\mathrm{I}}(\mathbf{k})} \end{split}$$

• Generic band crossings from tuning only two parameters! (Pancharatnam, Berry, ...)



EPs come in pairs and are generic in 2d, hence much more abundant than in the Hermitian case!

Numerical observations





• $E(\mathbf{k})$ is different than what one naively infers from $E^2(\mathbf{k})!$ $\operatorname{Re}[E(\mathbf{k})]$ $\operatorname{Im}[E(\mathbf{k})]$ $\operatorname{Im}[E(\mathbf{k})]$



- 2d bulk Fermi arcs!? V. Kozii and L. Fu, arXiv:1708.05841

Let's have a closer look: arcs

$$E(\mathbf{k}) = \pm \sqrt{\mathbf{d}_{\mathrm{R}}(\mathbf{k})^{2} - \mathbf{d}_{\mathrm{I}}(\mathbf{k})^{2} + 2i\mathbf{d}_{\mathrm{R}}(\mathbf{k}) \cdot \mathbf{d}_{\mathrm{I}}(\mathbf{k})}$$



Irremovable degeneracies; generic (d-1)-dimensional open nodal surfaces/arcs

Let's have a closer look: exceptional nature of nodes

• Any E = 0 solution with $\mathbf{d} \neq 0$ is exceptional, i.e. it corresponds to a defective Hamiltonian

• Rotate the basis such that
$$H = d(\sigma_x + i\sigma_y) = 2d \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
 Ok!

• EPs are non-analytical, "square roots of Weyl points"

$$E(\mathbf{k}) = \pm \sqrt{\mathbf{d}_{\mathrm{R}}(\mathbf{k})^{2} - \mathbf{d}_{\mathrm{I}}(\mathbf{k})^{2} + 2i\mathbf{d}_{\mathrm{R}}(\mathbf{k}) \cdot \mathbf{d}_{\mathrm{I}}(\mathbf{k})}$$



Splitting Weyl/Dirac points

• Minimal 2d model

$$H = k_x \sigma_x + k_y \sigma_y + i\epsilon \sigma_x$$

$$= \sum E = \pm \sqrt{k_x^2 + k_y^2 - \epsilon^2 + 2i\epsilon k_x}$$

$$\operatorname{Re}[E] = 0$$







"Braiding"

• Move around an exceptional point, track an eigenstate



- Cf. our simple example and non-Abelian braiding
 - But no obvious analogue of adiabatic transport...

We end up in the other
 eigenstate after one
 closed orbit!

Experimental observation of 2d bulk Fermi arcs

 Fermi arcs observed in photonic crystal slabs with gain and loss

H.Zhou, et. al. Science p. eaap9859 (2018)

• These experiments directly measure $\operatorname{Re}[E(\mathbf{k})]!$



Light scattering, iso-frequency contours vs. theoretical band structure

Material junctions?

• Example: 3d Topological insulator coupled to a ferromagnetic lead

$$H_{\rm NH} = H + \Sigma_L^r (\omega = 0)$$

- Surface theory + exact lead self energy:

$$\tilde{H} = \lambda (k_y \sigma_x - k_x \sigma_y) + \Sigma_L^r(0) - B \sigma_z \equiv \epsilon_0 + \mathbf{d} \cdot \boldsymbol{\sigma}$$

- Symmetry protected state promoted to a generic topological phase!
 - Sufficiently generic coupling needed
- Other NH topological states can be created similarly at the boundary of materials connected to leads

E.J. Bergholtz and J.C. Budich, Phys. Rev. Research 1,012003 (2019)





3d: exceptional rings,...

- Remember: $E(\mathbf{k}) = \pm \sqrt{\mathbf{d}_{\mathrm{R}}(\mathbf{k})^2 \mathbf{d}_{\mathrm{I}}(\mathbf{k})^2 + 2i\mathbf{d}_{\mathrm{R}}(\mathbf{k}) \cdot \mathbf{d}_{\mathrm{I}}(\mathbf{k})}$
- E=0 solutions form exceptional rings Y. Xu, S.-T. Wang, and L.-M. Duan, PRL 118, 045701 (2017)
- Think about this geometrically
 - Intersections between 2d surfaces
 - J. Carlström and E.J. Bergholtz, Phys. Rev. A 98, 042114 (2018)
- Leads to unusual open Fermi surfaces
 - Terminated by exceptional lines



Exceptional links and twisted "Fermi Ribbons"

 Exceptional links generated as generic intersections between more general 2d closed surfaces

J. Carlström and E.J. Bergholtz, Phys. Rev. A 98, 042114 (2018)



Leads to open "Fermi ribbons"







- Seifert surfaces, orientable

Generalization: Knotted non-Hermitian metals

J. Carlström, M. Stålhammar, J.C. Budich and E.J. Bergholtz, Phys. Rev. B 99, 161115 (2019)



- Two notions of topology combined a unique NH possibility
 - Hermitian generic line-like nodes occur in D=4, but in D>3 all knots are trivial!
- Short-range tight-binding models nearest neighbour for a nodal link and next nearest for a trefoli knot.
- Hyperbolic knots and links also possible...

M. Stålhammar, et. al., SciPost Phys. 7, 019 (2019)

Symmetries in non-Hermitian systems

- Specifically non-Hermitian symmetries
- D. Bernard and A. LeClair, arXiv:cond-mat/0110649 (2001) S. Lieu, arXiv:1807.03320
- Example $H = qH^{\dagger}q^{-1}, \quad q^{\dagger}q^{-1} = qq^{\dagger} = \mathbb{I}.$ "Pseudo hermiticity"
 - For 2-band models, pick $q=\sigma_x$
 - $d_x, d_0 \in \mathbb{R}, \qquad d_y, d_z \in i\mathbb{R}.$
 - Trivial in the Hermitian limit

- Generally
$$\mathbf{d}_R \cdot \mathbf{d}_I = 0$$
 \longrightarrow $E_{\pm} = \pm \sqrt{d_R^2 - d_I^2},$ $(d_0 = 0)$

Purely real or imaginary!

• \mathcal{PT} symmetric systems, popular in optics, work analogously



• 2d example

 $H = (2 - \cos k_x - \cos k_y)\sigma_x + i\sigma_z/4$

J.C. Budich, J. Carlström, F.K. Kunst and E.J. Bergholtz, in arXiv:1810.00914 + several subsequent postings...

Exceptional rings: Experiments

 Realised with coupled waveguides

Cerjan et. al. Nature Photonics 13, 623 (2019)



- Links and knots to come...
- Exotic new bulk physics but how about boundary states?

Minimal example 2: single band topology, large matrices and open boundaries

• Take home: Topology of complex energies & strong response to boundary conditions

Hatano-Nelson model

N. Hatano and D.R. Nelson Phys. Rev. Lett. 77, 570 (1996)

Single-band model with asymmetric hopping

- Complex dispersion relation

$$E_k = (J_L + J_R)\cos(k) + i(J_L - J_R)\sin(k)$$

 Winding number distinguishes different phases with respect to a point gap

$$w = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \mathrm{d}k \,\partial_k \ln E_k$$

- Phase transition at $|J_L| = |J_R|$



Hatano-Nelson with open boundaries

N. Hatano and D.R. Nelson Phys. Rev. Lett. 77, 570 (1996)

 Note 1: Adding a constant changes the winding number! —> no bulk boundary correspondence in the usual sense



 Note 2: The spectrum with open boundaries is completely different from the periodic system — states pile up at one of the boundaries!

$$H_{\text{open}} = \begin{pmatrix} 0 & J_R & 0 & 0 & 0 \\ J_L & 0 & J_R & 0 & 0 \\ 0 & J_L & 0 & \cdots & 0 \\ 0 & 0 & \vdots & \ddots & J_R \\ 0 & 0 & 0 & J_L & 0 \end{pmatrix}$$

- Single Jordan block and order N exceptional point a $J_L = 0$
- Extreme sensitivity to boundary conditions
- Actually, what, more precisely, do we mean by "states" above?

Focus 2: Anomalous bulk-boundary correspondence



F. K. Kunst, E. Edvardsson, J. C. Budich, and E. J. Bergholtz, Phys. Rev. Lett. 121, 026808 (2018) (Editors' Suggestion)

Alternative approach :

S.Yao, F. Song, and Z.Wang, Phys. Rev. Lett. 121, 136802 (2018) (Editors' Suggestion)

 Take home: Open and closed boundary conditions give very different physics — but cases can be understood and are experimentally relevant!

Basic observation (cf. also minimal example 2)

• Open and periodic energy spectra can be dramatically different!



- Literature filled with topological invariants calculated with periodic boundary conditions — but these do not generally dictate the presence of boundary modes!
- Need to consider the open system from the outset...

Non-Hermitian skin effect

• At the heart of the problem

Left and right eigenstates pile up at opposite sides

But their "product" does not

Biorthogonal quantum mechanics

Brody, J Phys. A: Math. Theor. 47, 035305 (2013)

• By definition we have

$$H|u_n^R\rangle = E_n|u_n^R\rangle$$
 and $H^{\dagger}|u_n^L\rangle = E_n^*|u_n^L\rangle$

 Away from exceptional points one can get a <u>complete orthonormal basis</u> by choosing

$$\langle u_n^L | u_m^R \rangle = \delta_{nm} \langle u_n^L | u_n^R \rangle$$

- Leading to

$$\sum_n \Pi_n = \mathbb{1}$$
 with $\Pi_n = \ket{u_n^R} ig \langle u_n^L |$

and $E_n = \langle u_n^L | H | u_n^R \rangle$ with $E_n \in \mathbb{C}$

• This provides the "product"...

A step back, again: Exact boundary states

Back to the non-Hermitian SSH chain

• Exact zero energy boundary states:

$$\begin{aligned} |\psi_R\rangle &= \mathcal{N}_R \sum_{m=1}^M r_R^m c_{\boldsymbol{A},m}^{\dagger} |0\rangle \qquad |\psi_L\rangle = \mathcal{N}_L \sum_{m=1}^M r_L^m c_{\boldsymbol{A},m}^{\dagger} |0\rangle \\ r_R &= -\frac{t_1 - \frac{\gamma}{2}}{t_2} \neq r_L = -\frac{t_1 + \frac{\gamma}{2}}{t_2} \end{aligned}$$

• Observation: when $|r_L^* r_R| = 1$ we have an exact zero energy biorthogonal bulk state!

$$\langle \Pi_m \rangle \equiv \langle \psi_L | \Pi_m | \psi_R \rangle \sim (r_L^* r_R)^m \quad \text{(now with } \Pi_m = |e_{A,m}\rangle \langle e_{A,m}| + |e_{B,m}\rangle \langle e_{B,m}|)$$
Not positive definite!

• Phase transitions and changes in zero-modes at $t_1 = \pm \sqrt{\frac{\gamma^2}{4} + t_2^2}, \pm \sqrt{\frac{\gamma^2}{4} - t_2^2}$?

Biorthogonal polarisation and boundary modes

- We construct a "biorthogonal polarisation", P , which is quantised and jumps precisely when $|r_L^\ast r_R|=1$

$$P \equiv 1 - \lim_{M \to \infty} \left\langle \psi_L \left| \frac{\sum_m m \Pi_m}{M} \right| \psi_R \right\rangle$$

- Predicts the correct phase transitions strikingly different from bulk invariants and also from indicators involving only right or left eigenstates!
 - Works also for non-solvable models and multiple boundary modes

It generalises directly: Non-Hermitian Chern insulators

Chern insulator phase diagram

Why does it work?

• Spectrum from left and right eigenvectors

$E_n = \langle u_n^L | H | u_n^R \rangle$

 Extended/delocalised biorthogonal states play the same role as extended states does in Hermitian models where the distinction between right and left is gone

Periodic vs. open boundary conditions

Inspired by: Xiong, Journal of Physics Communications 2, 035043 (2018)

- Crossover at exponentially small Γ

Intuitive from the perspective of the skin effect

Domain walls

• Physical mechanism: coupling ends via a Hermitian domain

- Both periodic and open system physics can be realised depending on the strength of the effective coupling!
 - Also tuneable geometrically and/or by Wannier function engineering

(In)stability of the spectra of large NH matrices

• Same thing, different perspective

- The spectrum of $H_{
m open}$ is stable to small perturbations like H_1 but not H_2

- Stability in math and physics literature quite different concepts

- See also poster by Loic Herviou

Experiments!

• Very recent experimental studies of the bulk-boundary correspondence

Topolectric circuits: Helbig et al., arXiv:1907.11562.

Mechanical/robotic system: Ghatak et al., arXiv:1907.11619.

Quantum walks: Xiao et al., arXiv:1907.12566

More inspiration...

Non-Hermitian sensing devices

- harnessing non-analytic dispersion relations

LETTER

Exceptional points enhance sensing in an optical microcavity

Weijian Chen¹, Şahin Kaya Özdemir¹, Guangming Zhao¹, Jan Wiersig² & Lan Yang¹

Nature volume 548, pages 192–196 (10 August 2017)

RESEARCH ARTICLE

TOPOLOGICAL PHOTONICS

Topological insulator laser: Theory

Gal Harari,¹⁺ Miguel A. Bandres,¹⁺ Yaakov Lumer,² Mikael C. Rechtsman,³ Y. D. Chong,⁴ Mercedeh Khajavikhan,⁵ Demetrios N. Christodoulides,⁵ Mordechai Segev¹

RESEARCH ARTICLE

TOPOLOGICAL PHOTONICS

Topological insulator laser: Experiments

Miguel A. Bandres, ¹* Steffen Wittek, ²* Gal Harari, ¹* Midya Parto, ² Jinhan Ren, ² Mordechai Segev, ¹† Demetrios N. Christodoulides, ²† Mercedeh Khajavikhan²†

Topological lasing mode

• Topological lasers?

- gain needed, previously thought to be in conflict with topological phases

Science 16 Mar 2018: Vol. 359, Issue 6381, eaar4005 *Science* 16 Mar 2018: Vol. 359, Issue 6381, eaar4003

Summary

Non-Hermiticity is relevant and interesting

- Exceptional degeneracies, square roots
- Gapless nodal phases theoretically far more abundant and conceptually rich than in the Hermitian realm
- Topology of complex energies & strong response to boundary conditions
- Open and closed boundary conditions give very different physics — but cases can be understood and are experimentally relevant!

Booming field with lot's of fun to discover...

See e.g. M.A. Bandres and M. Segev, Physics 11,96 (2018) and V. M. Martinez Alvarez, J. E. Barrios Vargas, M. Berdakin, and L. E. F. Foa Torres, Eur. Phys. J. Spec. Top. (2018)

Review together with Flore Kunst and Jan Budich to appear soon...

Flore Kunst, Elisabet Edvardsson, Marcus Stålhammar, and Johan Carlström in Stockholm

Jan Budich in Dresden