An invitation to topological semi-metals and their (non-linear) responses:

Lecture 1: • Weyl semimetals basics
• linear response and surface states

Lecture 2: • Non-linear optical responses. (in metals)
• Quantized non-linear response in Weyl semimetals.

interesting!
• quantized non-linear response of metals.!
(not so studied)

Lecture 1.

REFS: General: notes on WSM web: 1909.02983
Strain: Ilan / Grushin / Pikulin 1903.11088
Anomalies: both of the above., Landsteiner anomalous transport

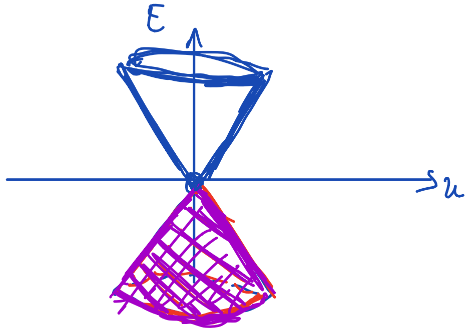
Topological semimetals (3D)

cont description $\longrightarrow$ lattice description

we are going to construct it in 3D:

think as real spin.

$$H_+ = + v_F \vec{k}\,\vec{\sigma} = v_F\left(k_x\sigma_x + k_y\sigma_y + k_z\sigma_z\right)$$



$E = \pm v_F|k|$

How can I gap out this state?

In a 2x2 subspace:

shifts the cone in energy space

$$H = H_+ + \vec{b}\,\vec{\sigma} + b_0\sigma_0$$

shifts the cone in momentum space

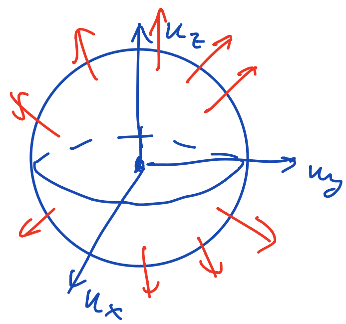
So no 2x2 matrix can gap it out!

Berry curvature around it?

$A_k^n = i\langle n|\partial_k|n\rangle$

$$\Omega_k^i = \left(\vec{\nabla}_k \times \vec{A}_k^n\right)_i = \frac{1}{2}\epsilon^{ijk}\hat{n}_k\left(\partial_{k_j}\hat{n}_k \times \partial_{k_l}\hat{n}_k\right) = +\frac{k^i}{2|k|^3}$$

$H = \vec{n}_k\cdot\vec{\sigma}$; $\hat{n}_k = \dfrac{\vec{n}_k}{|\vec{n}_k|}$



But Berry curvature must be periodic!

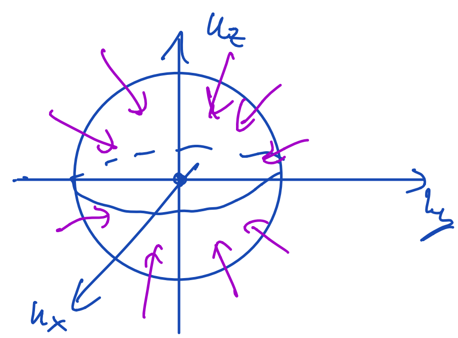
$$\frac{1}{2\pi}\int d\vec{S}\,\hat{\vec{\Omega}}^{1}_{n} = C \equiv \text{Monopole charge}$$

(chern number)

So there exists a second monopole of opp charge in the BZ. that acts as sink of Berry flux.

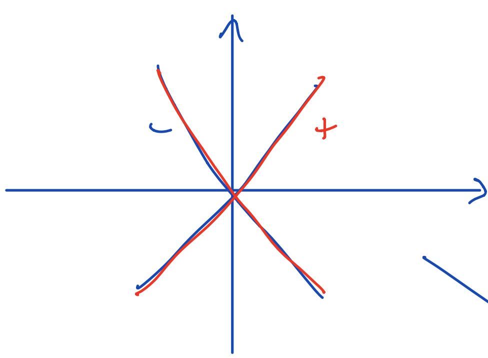
$$H_- = -v_F\,\vec{k}\cdot\vec{\sigma}$$

chirality.



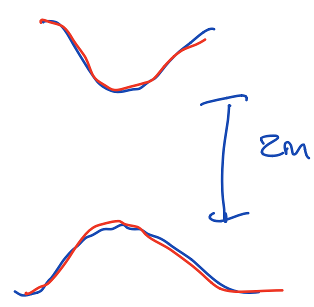
So minimal low energy model $H = \left(\begin{array}{c|c} H_+ & 0 \\ \hline 0 & H_- \end{array}\right)$

two decoupled chiralities.

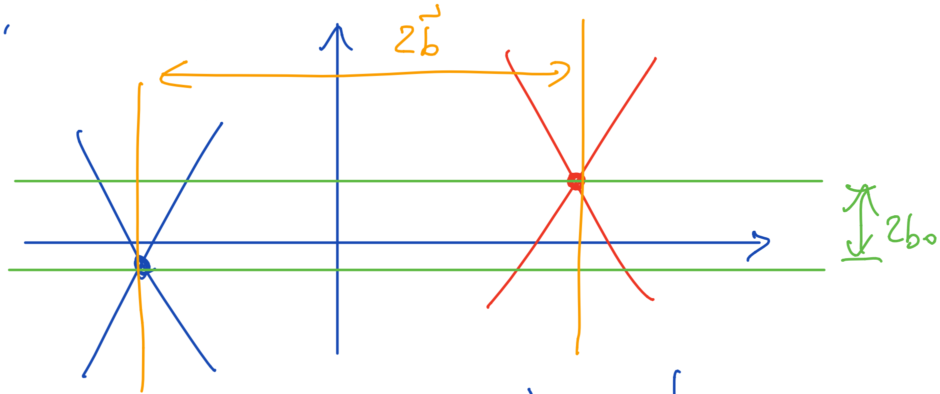


but they could talk, with off diagonal terms

$$\left(\begin{array}{c|c} H_+ & m \\ \hline m & H_- \end{array}\right)$$



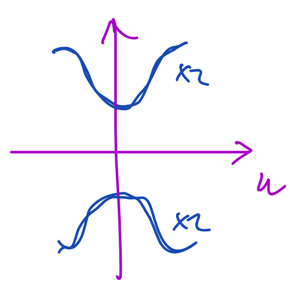
To avoid this, we separate chiralities in momentum space.



So $H = \left(\begin{array}{c|c} b_0 + (\vec{k}+\vec{b})\vec{\sigma} & m\sigma_0 \\ \hline m\sigma_0 & -b_0 - (\vec{k}-\vec{b})\vec{\sigma} \end{array}\right)$

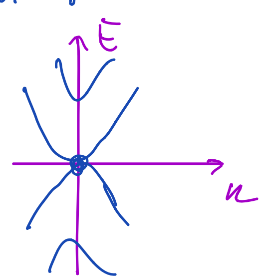
$b^2 \equiv b_0^2 - b^2$

for $m^2 >> -b^2$

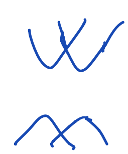


insulator.

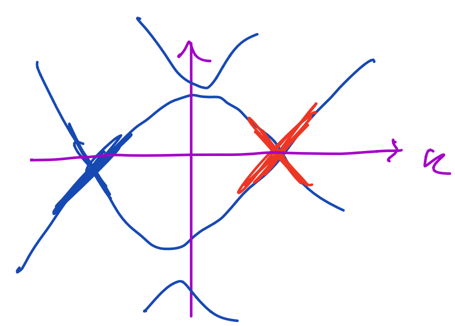
for $m = -b^2$



only b_0:



for $m << -b^2$



Weyl semimetal!

compact rep

$H = b_0\, \tau_z \otimes \sigma_0 + (\tau_z \vec{k} + \vec{b})\, \tau_z \otimes \vec{\sigma} + m\, \tau_x \otimes \sigma_0$

$\tau_i \otimes \sigma_i \equiv$ kronecker product

Symmetries:

TRS: $T H_k T^{-1} = H_{-k}$ (anti-unitary) inversion: $I H_k I^{-1} = H_{-k}$ (unitary)

(1) $H = \vec{b}\cdot\vec{\sigma}\otimes\tau_0$ breaks TRS. $T = \underbrace{\tau_0}_{\text{nothing for orbitals}}\underbrace{i\sigma_y}_{\text{spin rotation}} K$ ← complex conj.

$$H_k = b_0\,\sigma_0\,\tau_z + (\tau_z \vec{k} + \vec{b})\,\vec{\sigma}$$

$T = \tau_0 i\sigma_y K$ ↗ momentum

$$(\tau_z \vec{k} + \vec{b})\, T\vec{\sigma}T^{-1} = -(\tau_z \vec{k} + \vec{b})\,\vec{\sigma}$$

$\neq H_{-k}$ because of $\vec{b}$ term

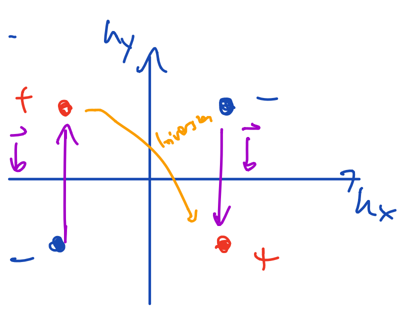
(2) $H = b_0\,\sigma_0\otimes\tau_z$ breaks inversion:

$$I = \tau_x \qquad I H_k I^{-1} = -b_0\,\sigma_0\,\tau_z - \tau_z \vec{k}\vec{\sigma} + \vec{b}\,\vec{\sigma}\otimes\tau_0$$

$\neq H_{-k}$ because of b_0 term.

Comments:

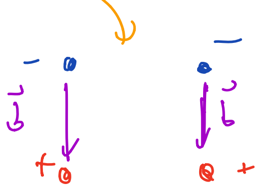
1) Minimal model with two Weyls breaks TRS
2) To have a Weyl model with TRS we have to break inversion.



TRS: $\begin{cases} \vec{\sigma} \to -\vec{\sigma} \\ k \to -k \\ + \to + \end{cases}$

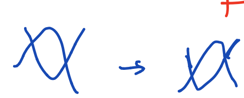
I: $\begin{cases} k \to -k \\ + \to - \end{cases}$

but this breaks TRS.



but inversion flips chirality

MIRROR$_x$: $\begin{cases} k_x \to -k_x \\ + \to - \end{cases}$



Responses from a lattice model.

Take a Chern insulator in (k_x, k_y) and vary the mass in the k_z direction [Q (who knows what a CI is?)]

$$H^{2D}_{CI} = t\sin k_x\,\sigma_x + t\sin k_y\,\sigma_y + (M - t\cos k_x - t\cos k_y)\,\sigma_z$$
$$\equiv \vec{n}_k \cdot \vec{\sigma}$$

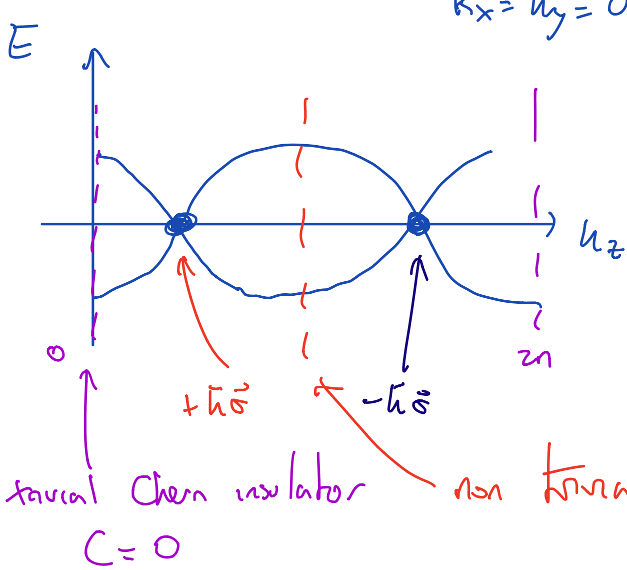
remember

C=0	-2	C=1	0	C=-1	2	C=0	M/t
(✱)		(✱)					
	gap closings @ (π,π)		$(0,\pi)$ $(\pi,0)$		$(0,0)$		

$M \to M(k_z) = M - \cos k_z$ with $M = -2$ $\begin{cases} M(0) = -3 \text{ (✱)} \\ M(\pi) = -1 \text{ (✱)} \end{cases}$

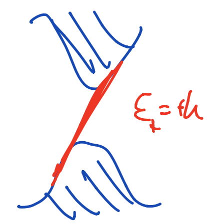
interpolates as a function of k_z from trivial to finite Chern number:

$k_x = k_y = 0$ $\qquad E = \pm|\vec{n}_k|$



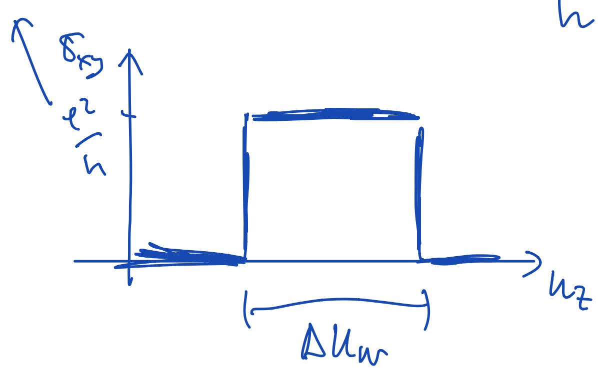
But we know 1) Chern insulator has a Hall conductivity $\sigma_{xy} = C\frac{e^2}{h}$

2) Has edge states.



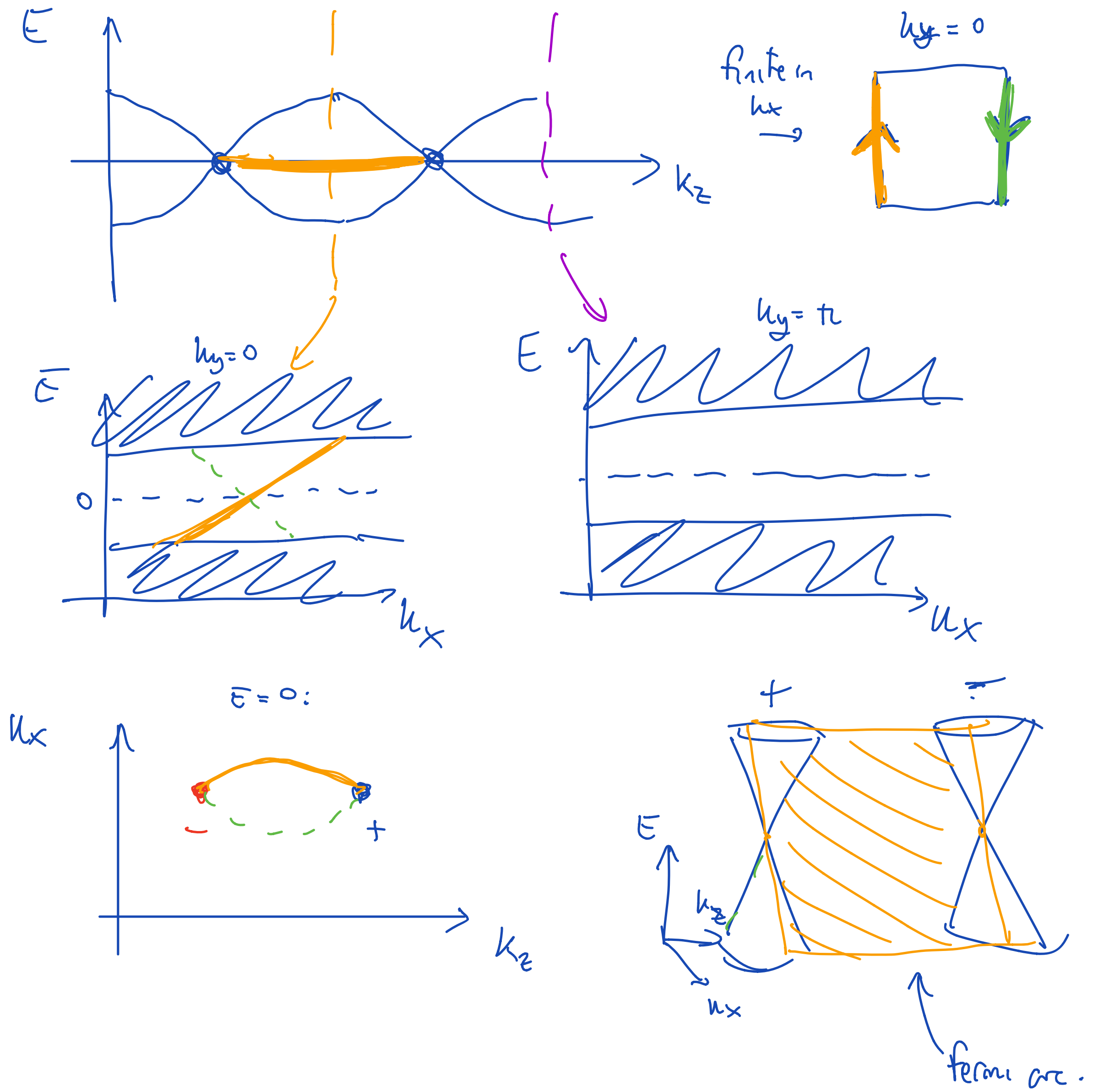
1) This means that the Hall conductivity of a Weyl semimetal:

$$\sigma_{xy} = \int_0^{2\pi} \sigma_{xy}(k_z)\, dk_z = \Delta k_W\, C_1 \frac{e^2}{h}$$



2) Between the Weyl points there exist the surface states of a Chern insulator for each k_z.

How do the bands look like for a finite system?



FIELD THEORY OF WEYL SEMIMETALS

Field Theory of Weyl Semimetals:

$$H = b_0 \overbrace{\tau_0 \otimes \sigma_0}^{\Gamma_5} + (\vec{k} + \tau_z \vec{b}) \overbrace{\tau_z \otimes \vec{\sigma}}^{\Gamma_i} + m \overbrace{\tau_x \otimes \sigma_0}^{\equiv \Gamma_0}$$

$$H = b_0 \Gamma_5 + \Gamma_i k_i + \Gamma_i \Gamma_5 b_i + m \Gamma_0$$

Legendre transform: $\mathcal{L} = \Psi_k^+ (\omega - H_k) \Psi_k$

to copy from hep: $\bar{\Psi} = \Psi^+ \Gamma_0$ and use that $\Gamma_0^2 = 1$:

$$\Rightarrow \quad \mathcal{L} = \underbrace{\Psi^+ \Gamma_0}_{\bar{\Psi}} (\Gamma_0 \omega - \Gamma_0 H_k) \Psi$$

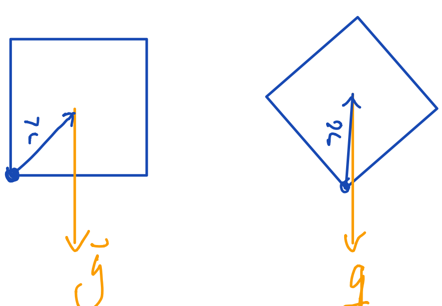
$$S = \int d^4k \, \bar{\Psi}_k (\gamma_\mu k^\mu - m + b_\mu \gamma^\mu \gamma_5) \Psi_k \quad \left\{ \begin{array}{l} \gamma_0 = \Gamma_0 \\ \gamma_i = \Gamma_0 \Gamma_i \\ \gamma_5 = \Gamma_5 \end{array} \right.$$

minimal substitution: $k^\mu \to k^\mu - e A^\mu$

$$S_{LBQED} = \int d^4k \, \bar{\Psi}_k (\gamma_\mu (k^\mu - e A^\mu + b_\mu \gamma_5) - m) \Psi_k$$

b_μ: 1) selects a prefered direction in space-time
2) you are not allowed to rotate it
since it is a parameter of the Hamiltonian
i.e. an expectation value. (e.g. magnetization)

e.g. Think of a box in a constant grav. field:



you can rotate coord system but if you don't rotate the vectors it is not Lorentz invariant = "particle" Lorentz transformation.

We notice that b_μ is like a field that distinguishes chirality because when $m=0$:

$$\gamma_5 \Psi = \pm \Psi$$

is this the most general theory? NO!
in a 4x4 space we have 16 matrices:

$$\underbrace{\mathbb{1},\ \gamma_0,\ \gamma_i,\ \gamma_5}_{6},\ \underbrace{\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]}_{6},\ \underbrace{\gamma_\mu \gamma_5}_{4}$$

So the most general Lorentz Breaking action is:

$$S = \int d^4k\ \bar{\Psi}_k \left(\Gamma_\mu \left(k^\mu - eA^\mu + b^\mu \gamma_5 + a^\mu \right) + H_{\mu\nu}\sigma^{\mu\nu} - m + m_5 \gamma_5 \right) \Psi$$

where we also added something in between γ_μ and k^μ

$$\Gamma^\mu \equiv \gamma^\mu + \overbrace{\underbrace{M^\mu{}_\nu}_{\text{anisotropic Fermi velocities}} \gamma^\nu + d^\mu{}_\nu \gamma^\nu \gamma_5}^{\text{Lorentz breaking.}} + \underbrace{e^\mu + f^\mu \gamma_5 + g^{\mu\nu\lambda}\sigma_{\nu\lambda}}_{\text{CPT breaking.}}$$

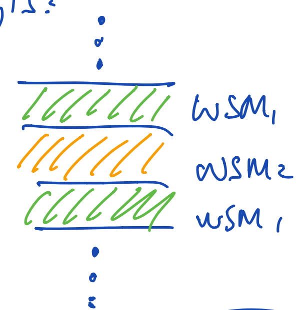
e.g. e^μ can give you tilted Weyls.
$H_{\mu\nu}$ can give you nodal lines.
Figure out the rest!! $\longrightarrow$ papers and papers.

People have focused mostly on the $\mu=0$ b^μ only kind.

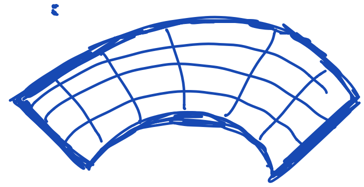
$$S=\int d^4x\; \bar{\Psi}_u\, \gamma_\mu \left(k^\mu - eA^\mu + b^\mu \gamma_5\right) \Psi_u$$

Recent interest in inhomogeneous Weyls:

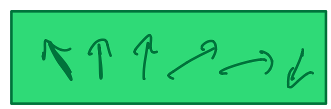
(A) Heterostructures



(B) Inhomogeneous strain



(C) Inhomogeneous magnetization



Turns out (see Ilan / Pikulin / Grushin review) all can be modelled by promoting (to a first approx)

$$b^\mu \longrightarrow b^\mu(x,t)$$

Now $b^\mu(x,t)$ is more like a field:

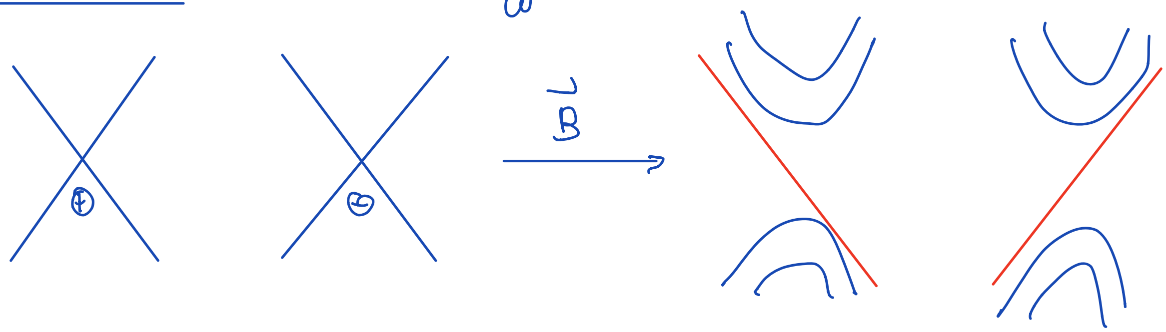
$$\vec{B}_5 = \vec{\nabla}\times\vec{b}$$

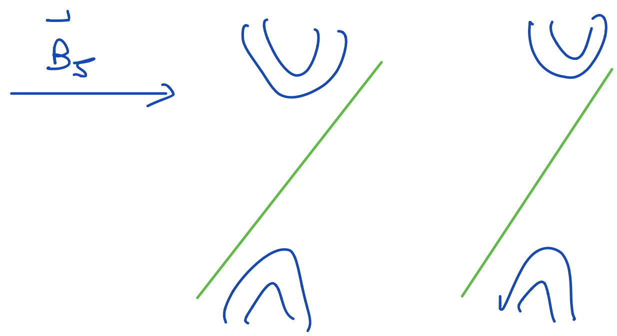
$$\vec{E}_5 = -\vec{\nabla} b_0 - \partial_t \vec{b}$$

Many open problems: quantum oscillations?
anomalies?
disorder?
surface states?

Anomalies:

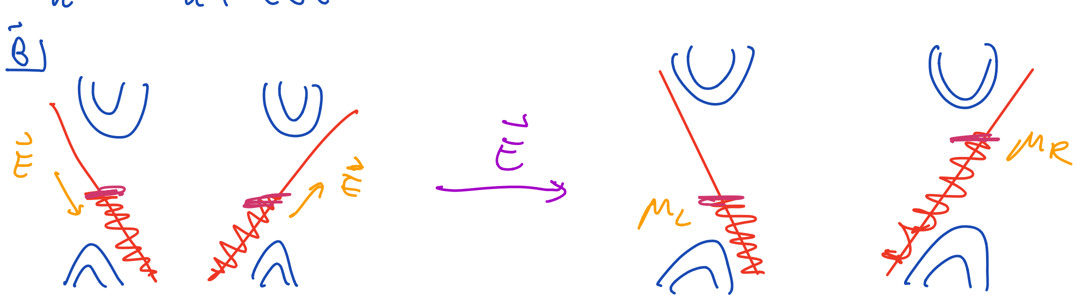
What is the effect of a $\vec{B}$ and a $\vec{B}_5$ field?





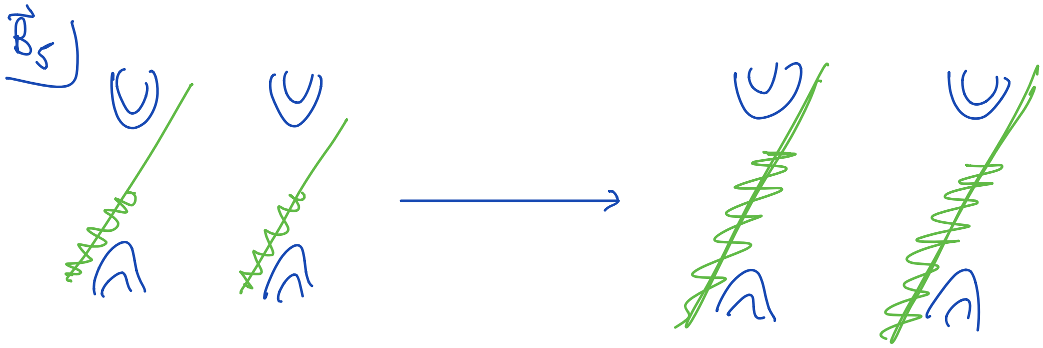
Electric field:

$\vec{k} \to \vec{k} + e\vec{E}t$



$\mu_{L/R} \propto n_{L/R}$

$$\partial_t (n_L - n_R) \propto \vec{E} \cdot \vec{B}$$ ← Landau level deg.



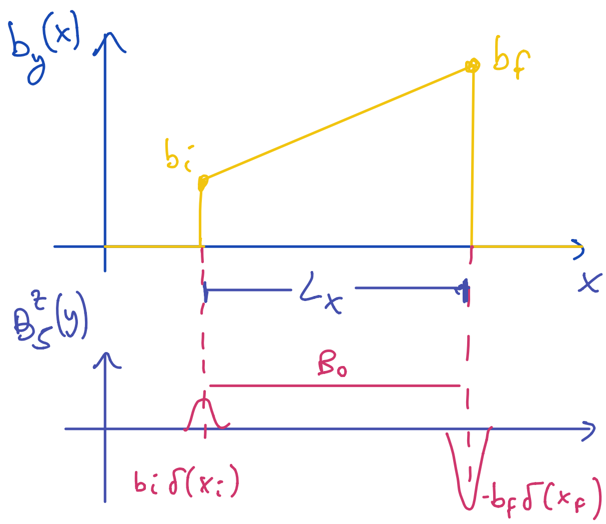
$$\partial_t (n_L + n_R) \propto \vec{E} \cdot \vec{B}_5 \quad !$$

This means that in the presence of dual and real e.m. fields neither the vector nor the chiral current are conserved:

$$\partial_\mu j^\mu_5 = c\,(\vec{E}\cdot\vec{B} + \vec{E}_5 \vec{B}_5) \qquad c = \frac{1}{2\pi^2}$$

$$\partial_\mu j^\mu = c\,(\vec{E}_5 \vec{B} + \vec{B}_5 \cdot \vec{E}) \qquad \text{(in horrible natural units)}$$

How can this be? We forgot to add the boundary. B_5 cannot be constant in space.



$B_5^z = \text{const} = B_0$

$\Rightarrow \; b(x,t) = b_y x \,\hat{y}$

but in reality:

$$b_y(x) = \begin{cases} 0 & x < x_i \\ \frac{b_f - b_i}{x_f - x_i}(x - x_i) + b_i & x_i \leq x \leq x_f \\ 0 & x > x_f \end{cases}$$

In such a way that $\int_{-\infty}^{\infty} dx\, B_5^z(x) = 0$.

Since $\int_{-\infty}^{\infty} dx\, B_5^z(x) = \int_{-\infty}^{\infty} dx\, \partial_x b_y =$

$$= \int_{-\infty}^{\infty} dx \left[b_i \delta(x_i) - b_f \delta(x_f) \right) + \int_{x_i}^{x_f} \frac{b_f - b_i}{L_x} dx$$

$$= 0$$

This is true for any B_5 profile so the edge states always compensate the bulk current response!

You can convince yourself using a lattice model.

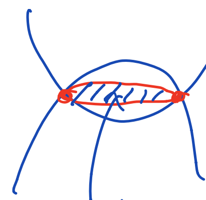
The real anomaly equations are:

$$\begin{cases} \partial_\mu j_5^\mu = \frac{1}{2\pi^2} \left(\vec{E} \cdot \vec{B} + \frac{1}{3} \vec{E}_5 \cdot \vec{B}_5 \right) \\ \partial_\mu j^\mu = 0 \end{cases}$$

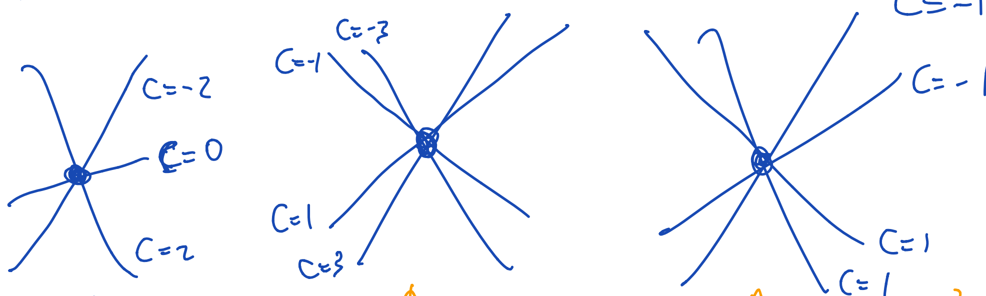
If you add $m(x)$ you can interpolate between Topological insulators and Weyl semimetals! $g^{\mu\nu} \to$ gravity etc.

Other (metallic beasts)

1.) Line node semimetals



2.) Multifold fermions:



3 fold

4-fold

Materials

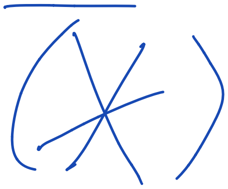
RhSi

CoSi

AlPt

$H=\vec{k}\cdot\vec{S}_1$ $\quad$ $H=\vec{k}\cdot\vec{S}_{3/2}$ $\quad$ $(H=\vec{k}\cdot\vec{\sigma})\times 2$

6 fold:



$\times 2$

8 fold no chirality.