Topological states in PEPS

Didier Poilblanc

LABORATOIRE DE PHYSIQUE THÉORIQUE TOULOUSE

- Topological spin liquids and canonical examples
- PEPS tensor network and algorithms
- Constructing simple topological PEPS spin liquids
- Chiral SL in a simple quantum spin model

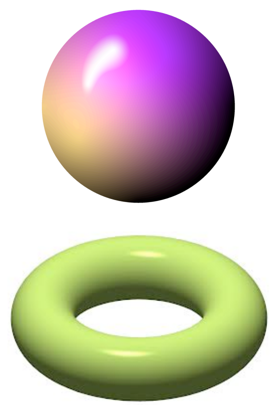
Exotic «spin liquids» beyond the «order parameter» paradigm

* no spontaneous broken symmetry
* no local order
* Topological order

  - Do they exist in materials ?
    in simple models ?

  - How to detect them ?

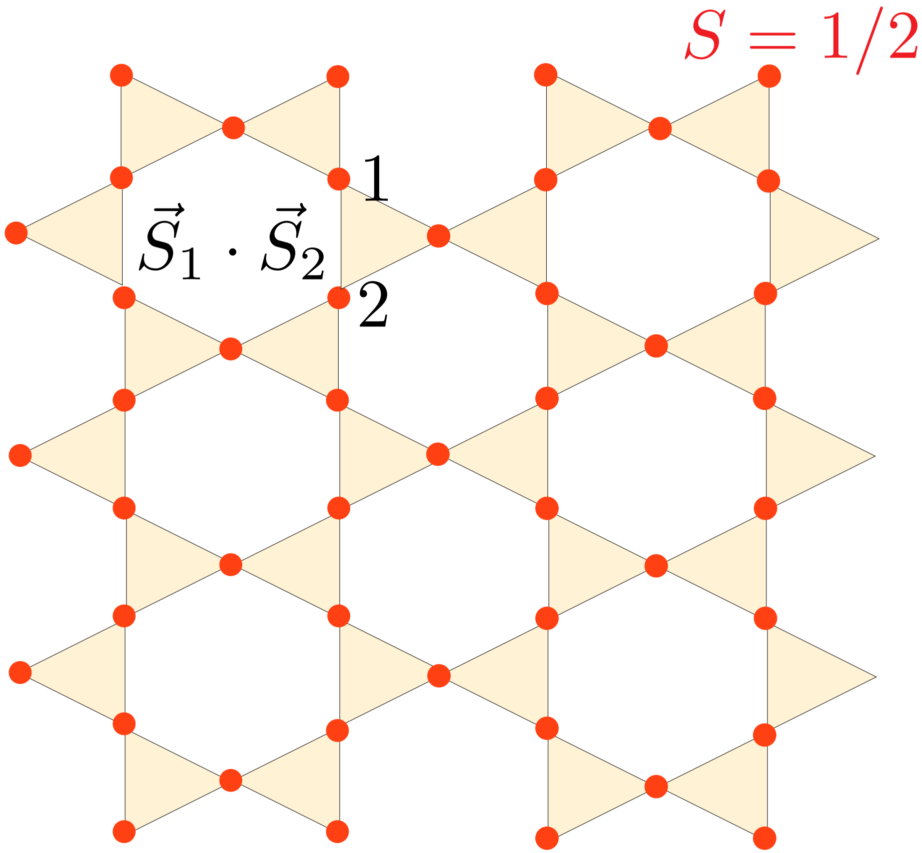
X. G. Wen



GS degeneracy (depends on **topology** of space)

Topological order can also be detected by entanglement measures !

Spin-1/2 Heisenberg QAF on the Kagome lattice

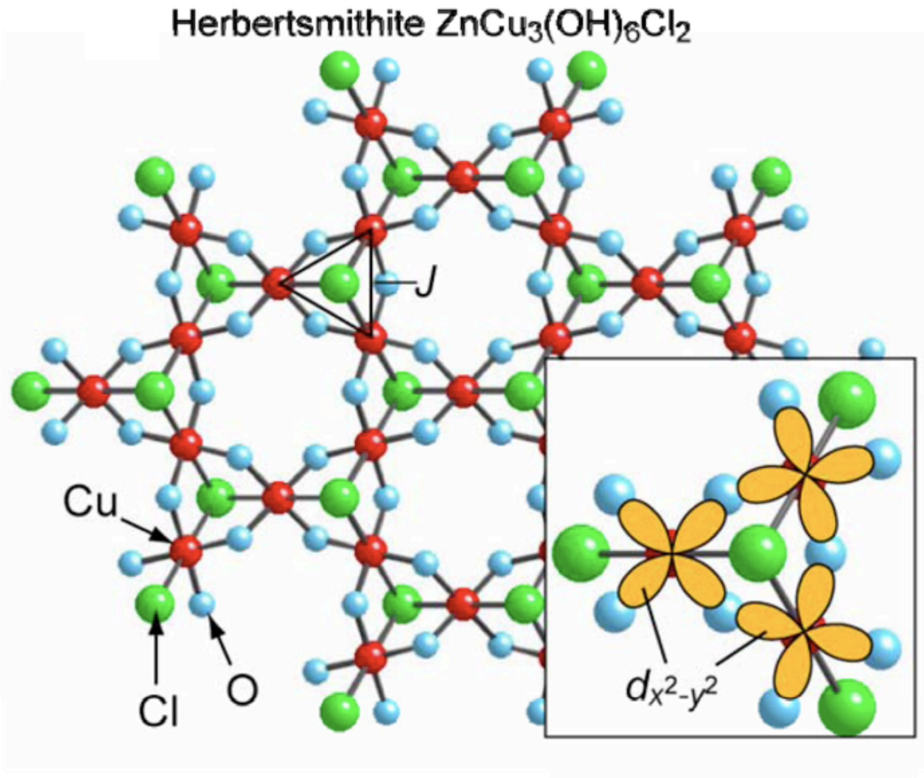


$$H = \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$

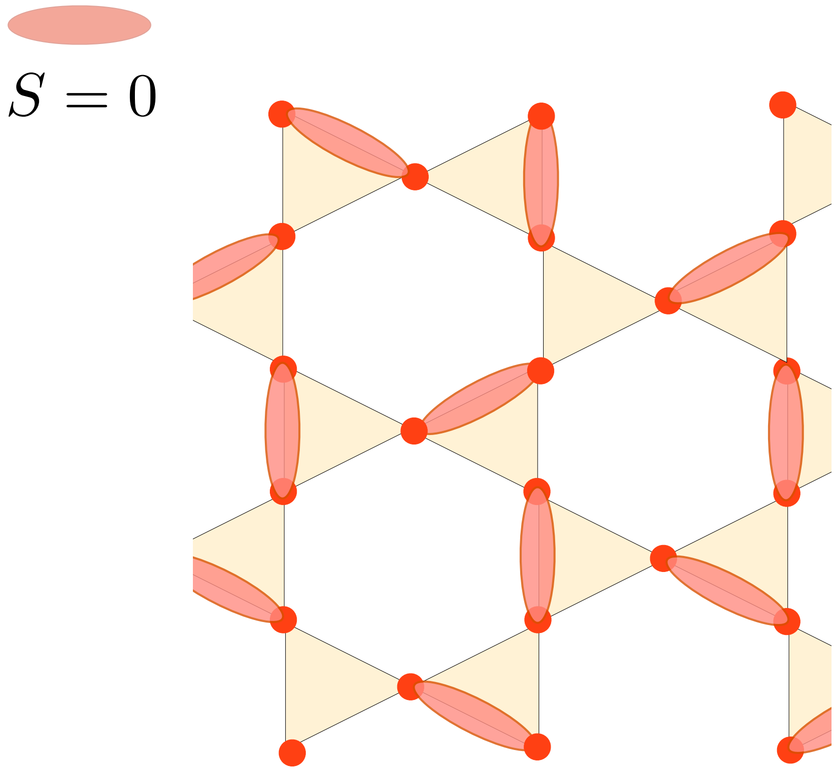
Best candidate for "spin liquid" behavior

Materials realizing a Kagome *"spin liquid"*

e.g. experiments by C. Broholm (MIT), Z. Hiroi (ISSP), P. Mendels (Orsay)...



A «toy» "non-trivial" spin liquid: the RVB spin liquid



Equal-weight superposition
of NN singlet coverings

spin-1/2 RVB

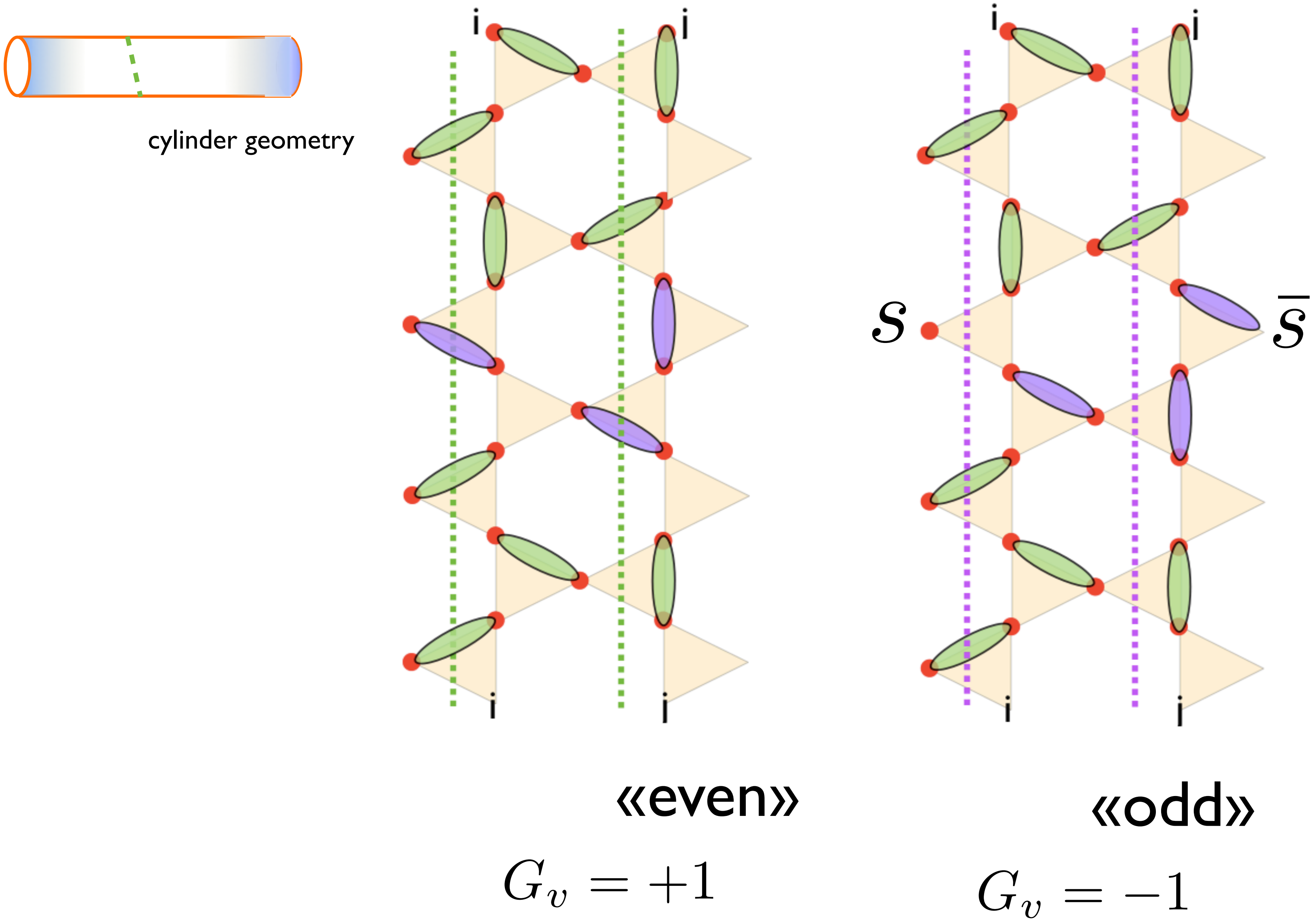
P. Fazekas and P.W. Anderson
Philosophical Magazine **30**, 423-440 (1974)

Hasting-Oshikawa-Lieb-Schulz-Mattis theorem

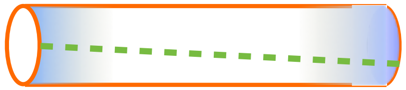
Rules out gapped SL with unique GS

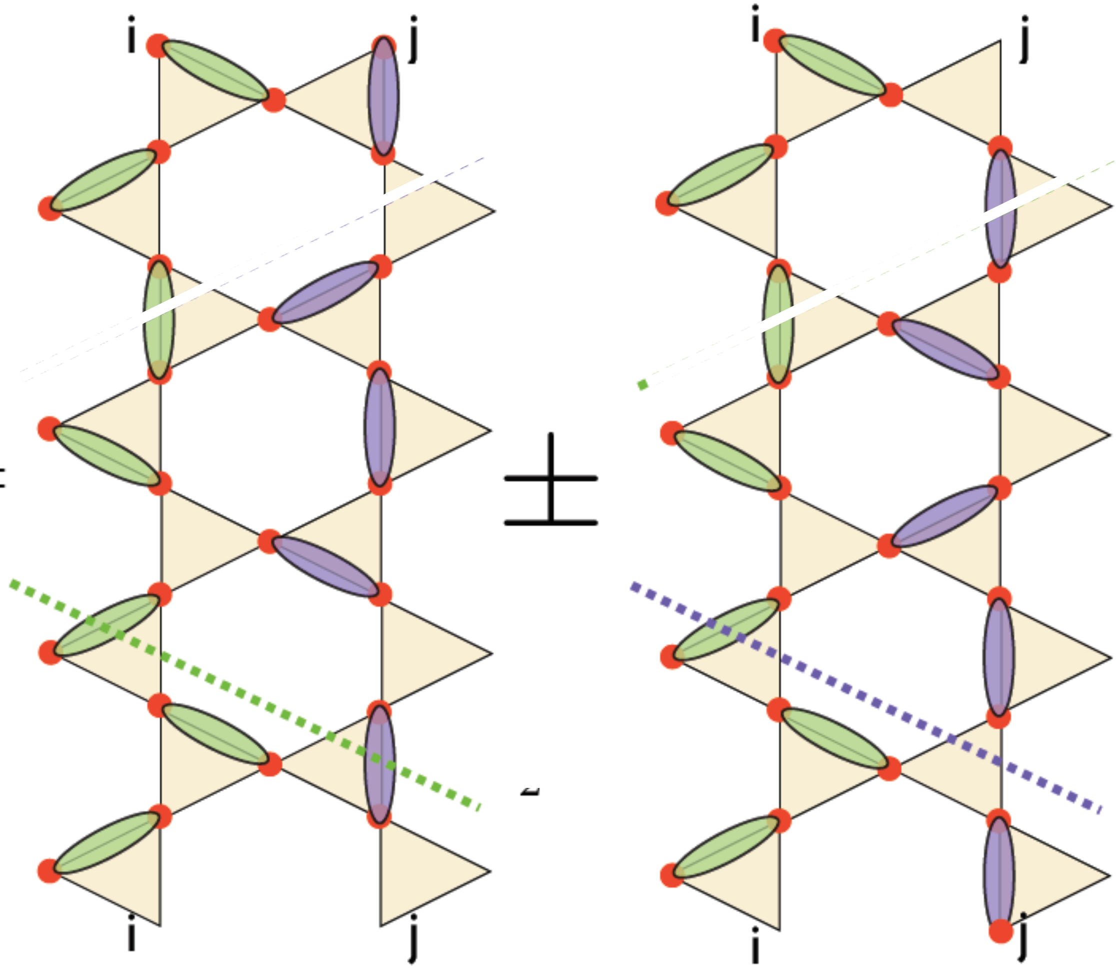
Topological liquid ?

Topological sectors



Eigenstates of a «Wilson loop» operator



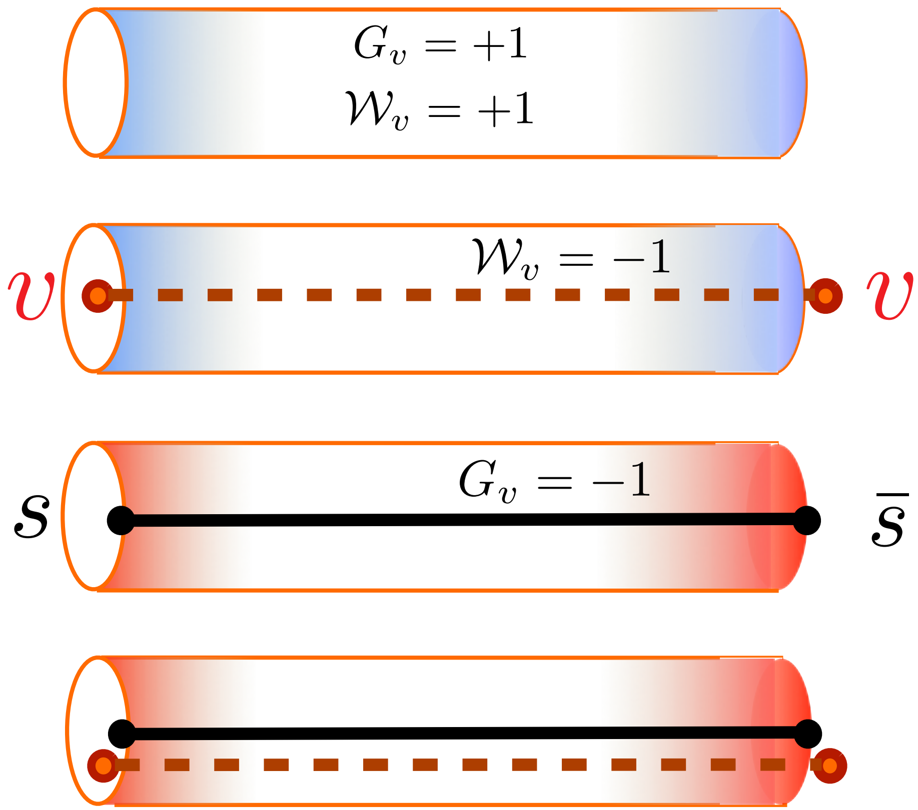


$$\Psi_{\rm RVB}^{\pm} = \quad \pm$$

$+$ no vison flux: $w_v = +1$

$-$ $\mathbb{Z}_2$ vison flux: $w_v = -1$

$\mathbb{Z}_2$ spin liquid :
topological GS inserting «spinons» and «visons»



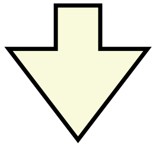
same class as
Kitaev's Toric Code
(fixed point $\xi = 0$)

All GS indistinguishable by local measurements

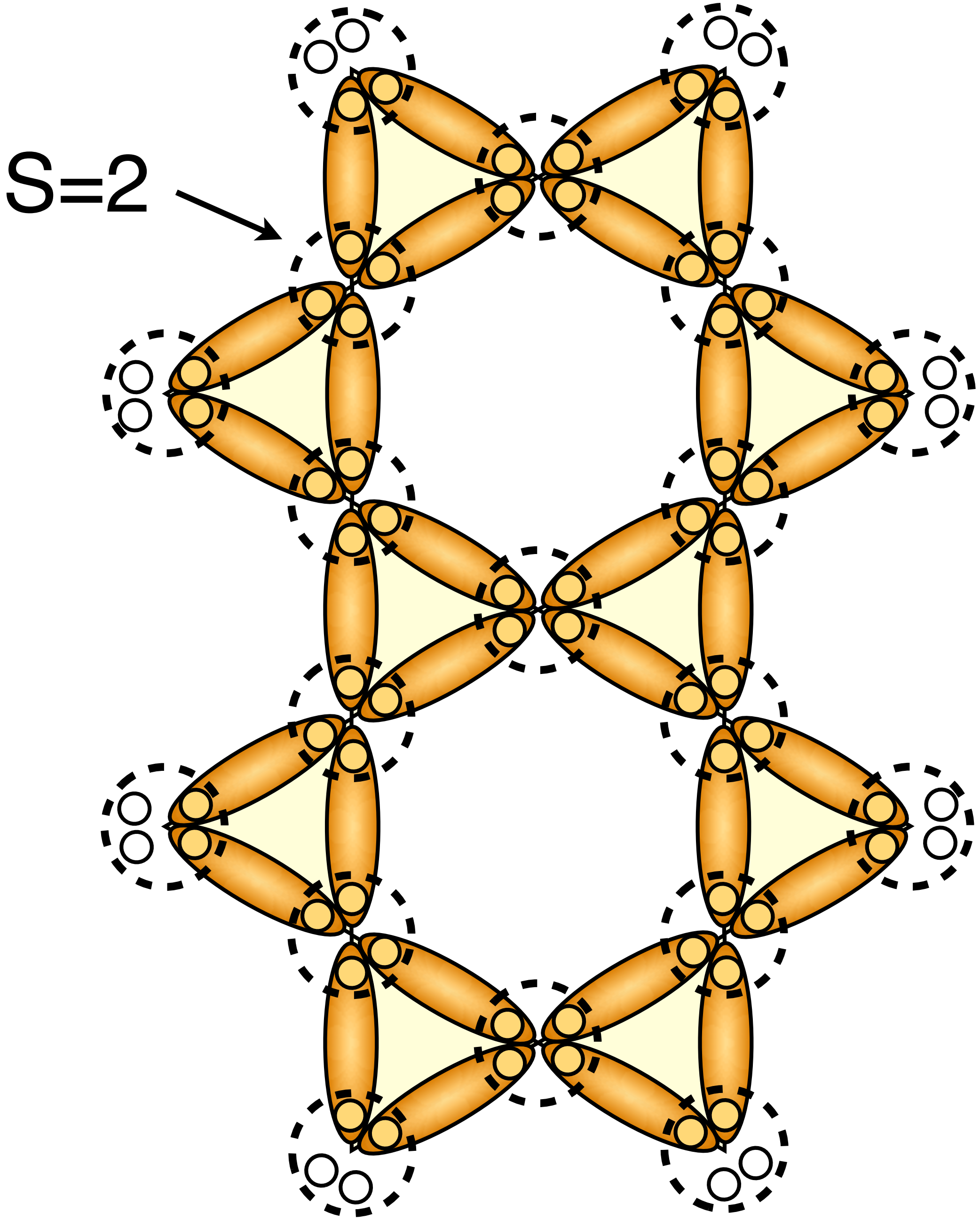
... vs «Trivial» insulator

$$H_{\text{AKLT}} = \sum_{\langle i,j \rangle} P_{ij}^{\mathbf{S}_i + \mathbf{S}_j = 4}$$

The Affleck-Kennedy-Lieb-Tasaki spin liquid

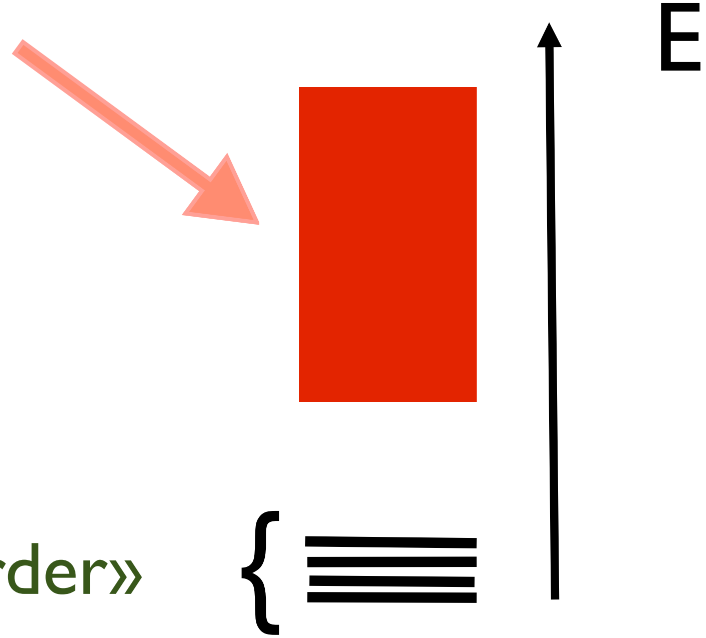


The GS is unique !



The many-body spectrum of a topological liquid

Excitations are fractional (created by pairs)



Degeneracy from «topological order»

How to capture topological order ?

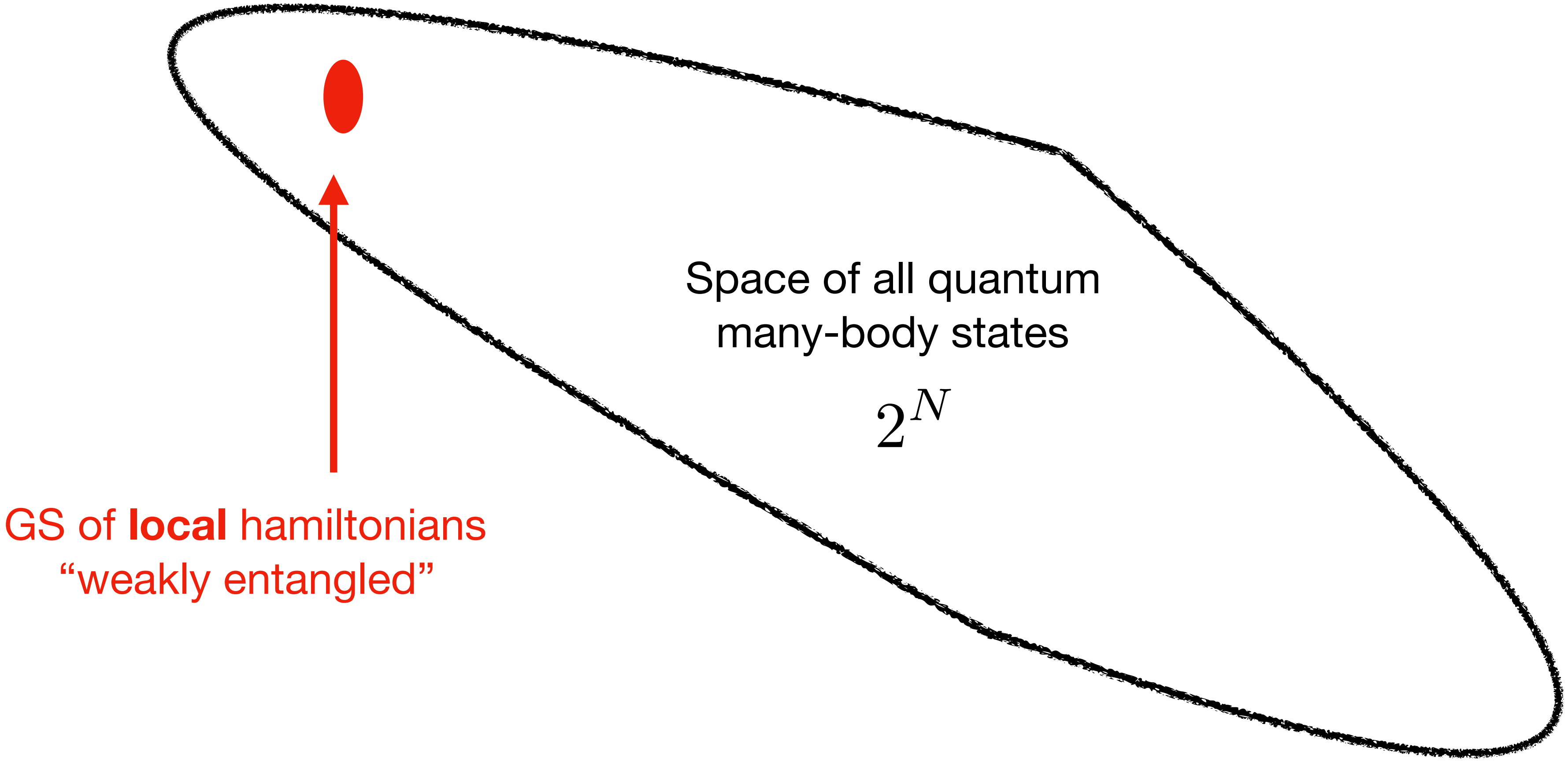
Topological order originates from **LR entanglement**

↓

Use new tools borrowed from quantum information based on the concept of entanglement

↓

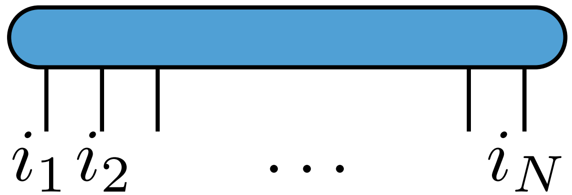
Tensor network methods


Space of all quantum many-body states
2^N
GS of **local** hamiltonians
“weakly entangled”

Tensor networks ansatze

G. Vidal
I. Cirac
D. Perez-Garcia
F. Verstraete

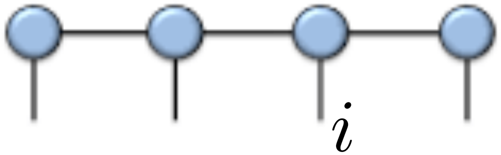
$$|\Psi\rangle = \sum C_{i_1, i_2, \ldots, i_N} |i_1, i_2, \ldots, i_N\rangle \qquad i_k = 1, \cdots, d$$

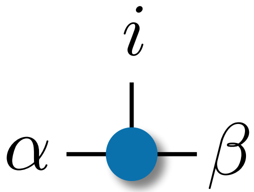


rank-N tensor with d^N elements

(d=2 for spin-1/2)

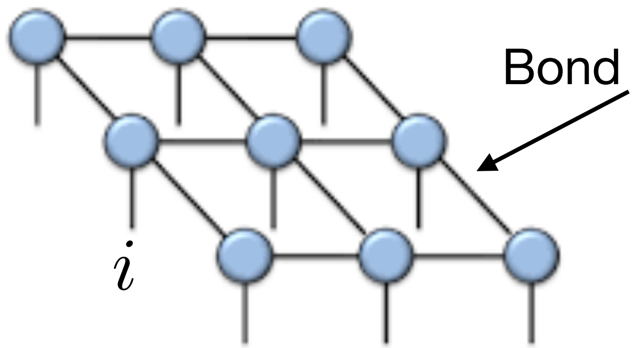
1D: Matrix product state (MPS)



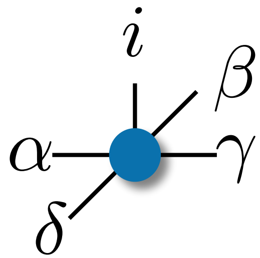


$$A^{(i)} \;:\; D \times D \text{ matrix}$$

(modern formulation of DMRG)



2D: Projected entangled pair state (PEPS)

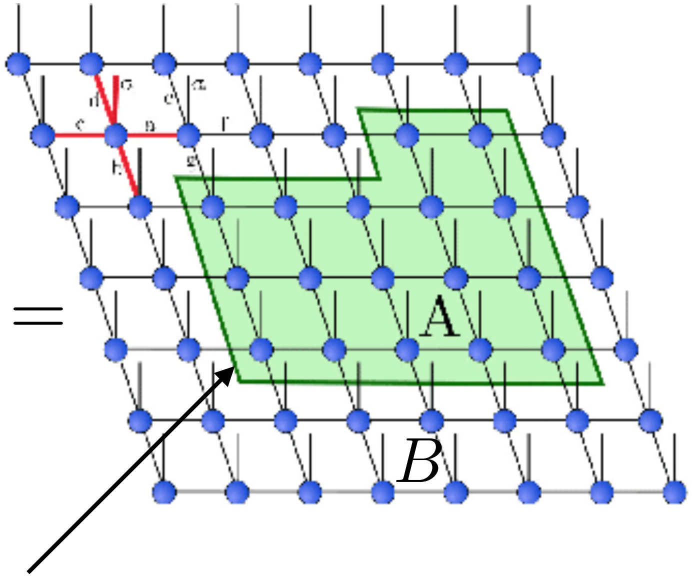


$D^{\otimes z}$ tensor

Area law

$$|\Psi\rangle \in \mathcal{E}_A \otimes \mathcal{E}_B$$

$$|\Psi\rangle =$$



perimeter L

$$\rho = |\Psi\rangle\langle\Psi| \qquad \text{projector}$$

Reduced density matrix :

P. Dirac (1930)

$$\rho_A = \sum_j \langle j|_B \left(|\Psi\rangle\langle\Psi|\right) |j\rangle_B = \mathrm{Tr}_B\, \rho$$

Entanglement entropy :

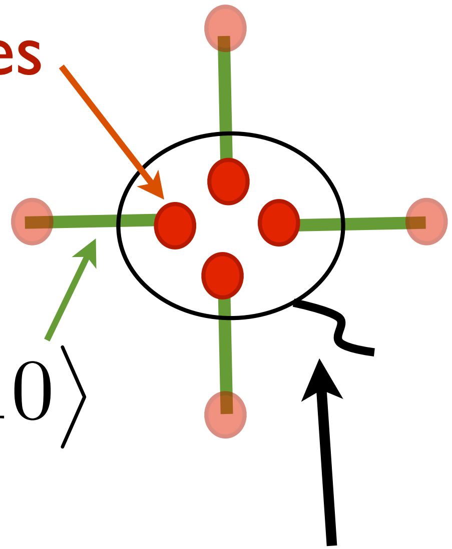
$$S_{\text{ent}} = -\mathrm{Tr}(\rho_A \ln \rho_A)$$

$$S_{\text{ent}} \leq D \times L$$

Can control the entanglement
via the bond dimension D!

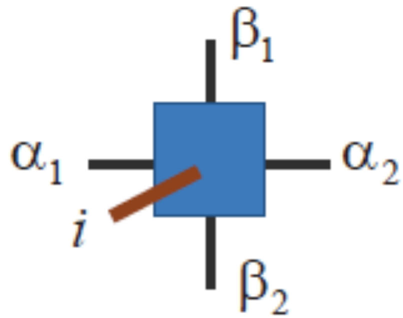
AKLT state can be represented as a Projected Entangled Pair State (PEPS) :

D=2 spin-1/2 virtual states



$$|\mathcal{S}\rangle = |01\rangle - |10\rangle$$

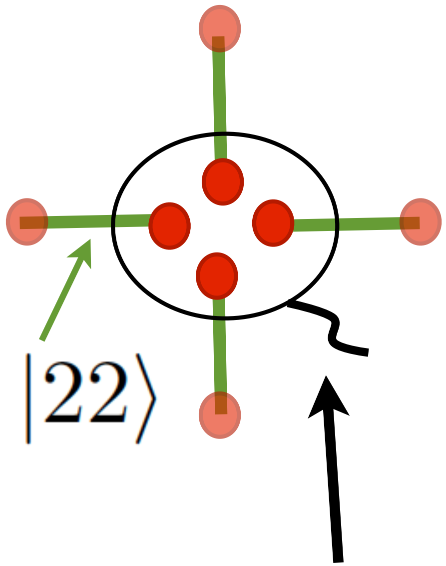
Project onto **physical** subspace $d = 2S + 1$:



local tensor

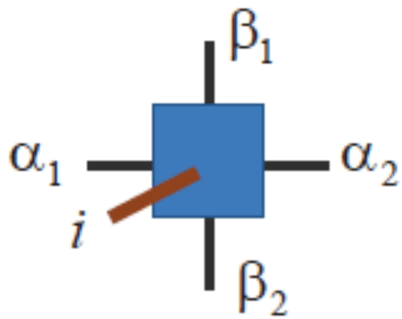
The spin-1/2 RVB can also be written as a PEPS !

virtual states: $1/2 \oplus 0$ (D=3)



$$|\mathcal{S}\rangle = |01\rangle - |10\rangle + |22\rangle$$

Project onto physical subspace S=1/2 (d=2)

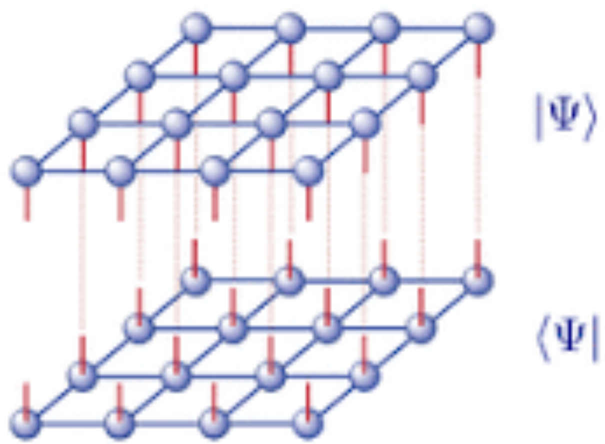
⇨ 

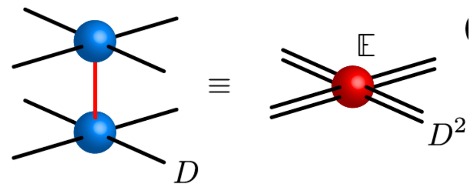
PEPS tensor

Z_2 gauge symmetry !

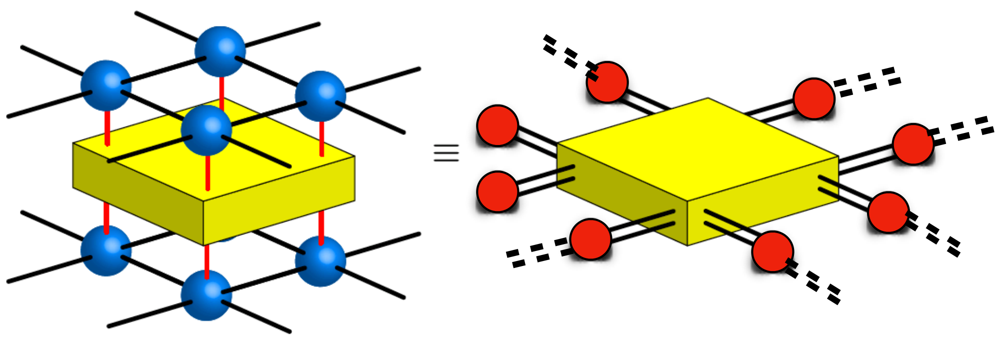
Computing observables

$$\langle \Psi | \Psi \rangle =$$





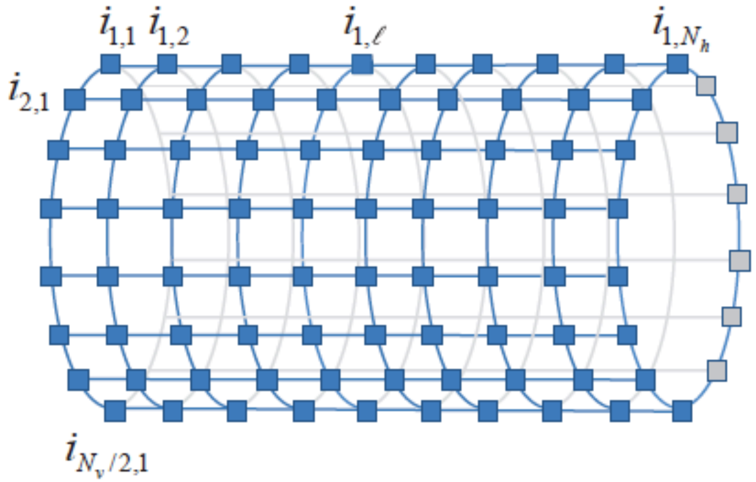
$$\langle \Psi | O_{1234} | \Psi \rangle =$$

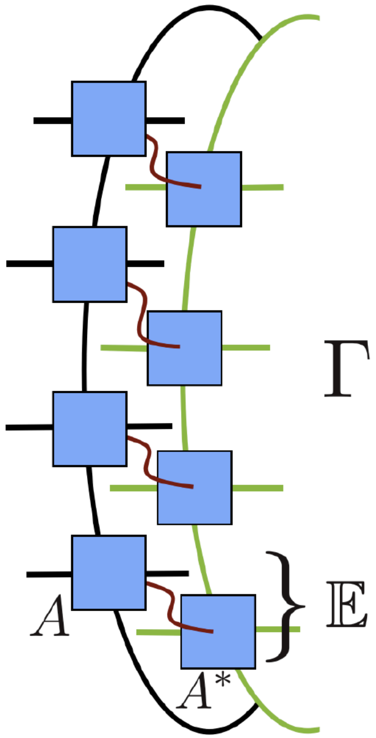


How to deal with infinite system ?

1) finite size scaling : finite PEPS
2) real space RG : iPEPS

Build «double layer» tensor network by contracting physical variables





$$\langle \Psi | \Psi \rangle = V_{\text{left}} \Gamma^{N_h} V_{\text{right}}$$

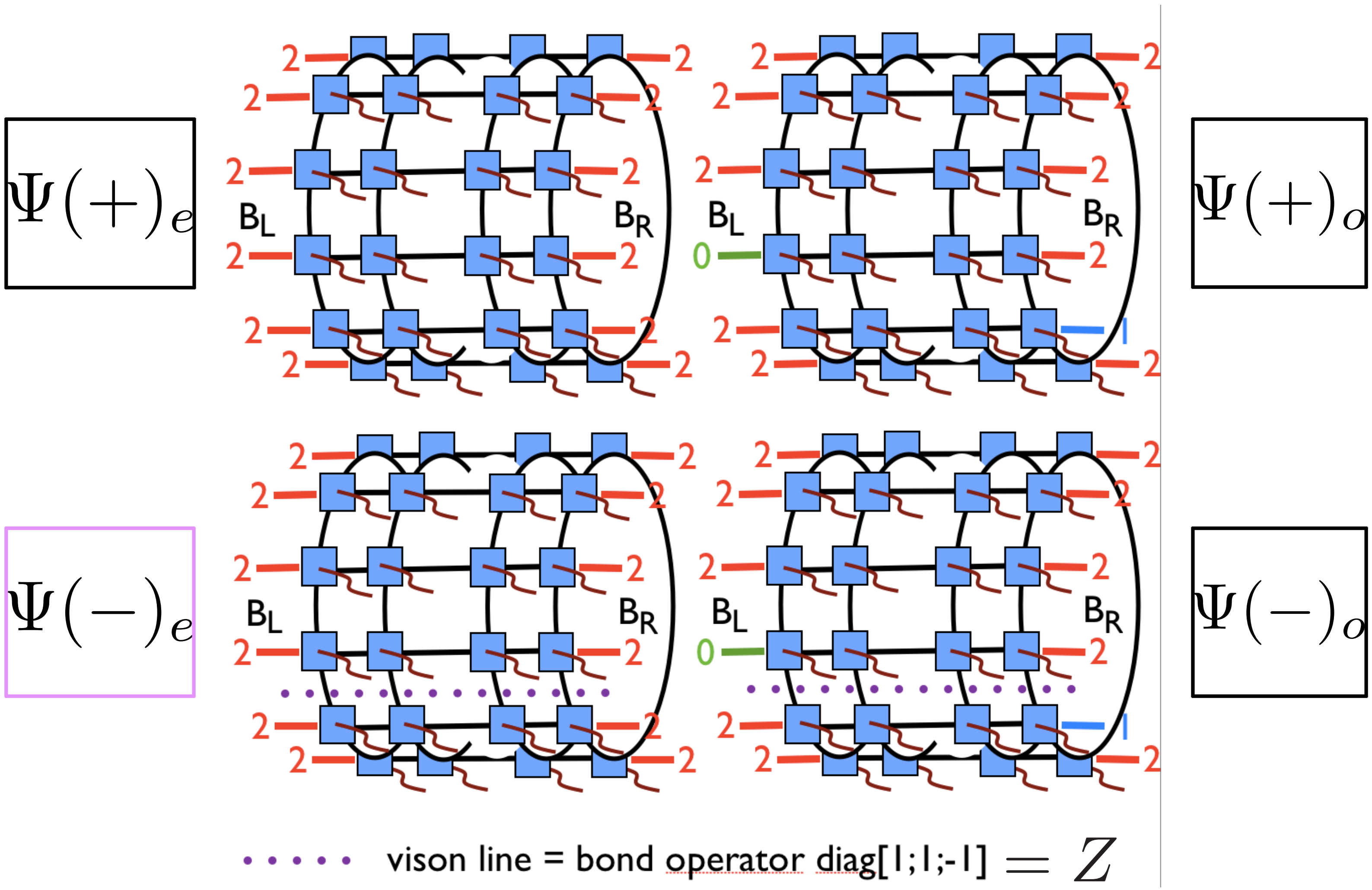
«Transfer Matrix»

$D^{2N_v} \times D^{2N_v}$ matrix

Iterate product of TM's to build infinite cylinder

if D small enough exact contractions possible…

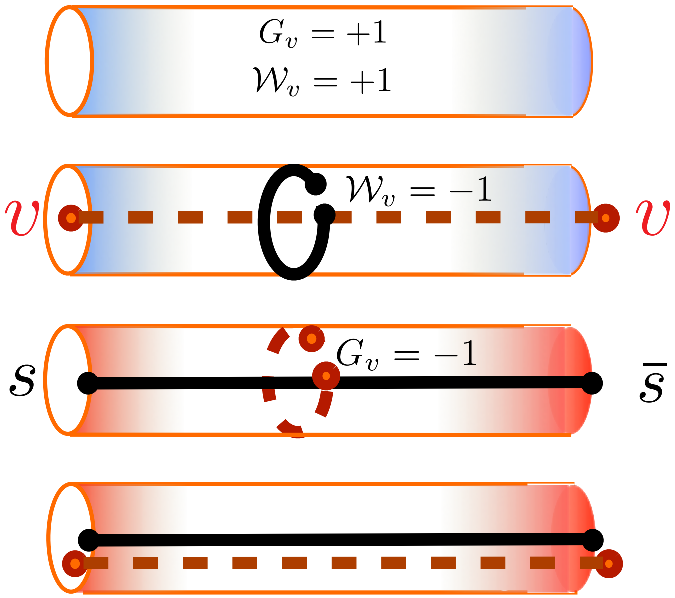
Easy formalism to construct all topological GS !



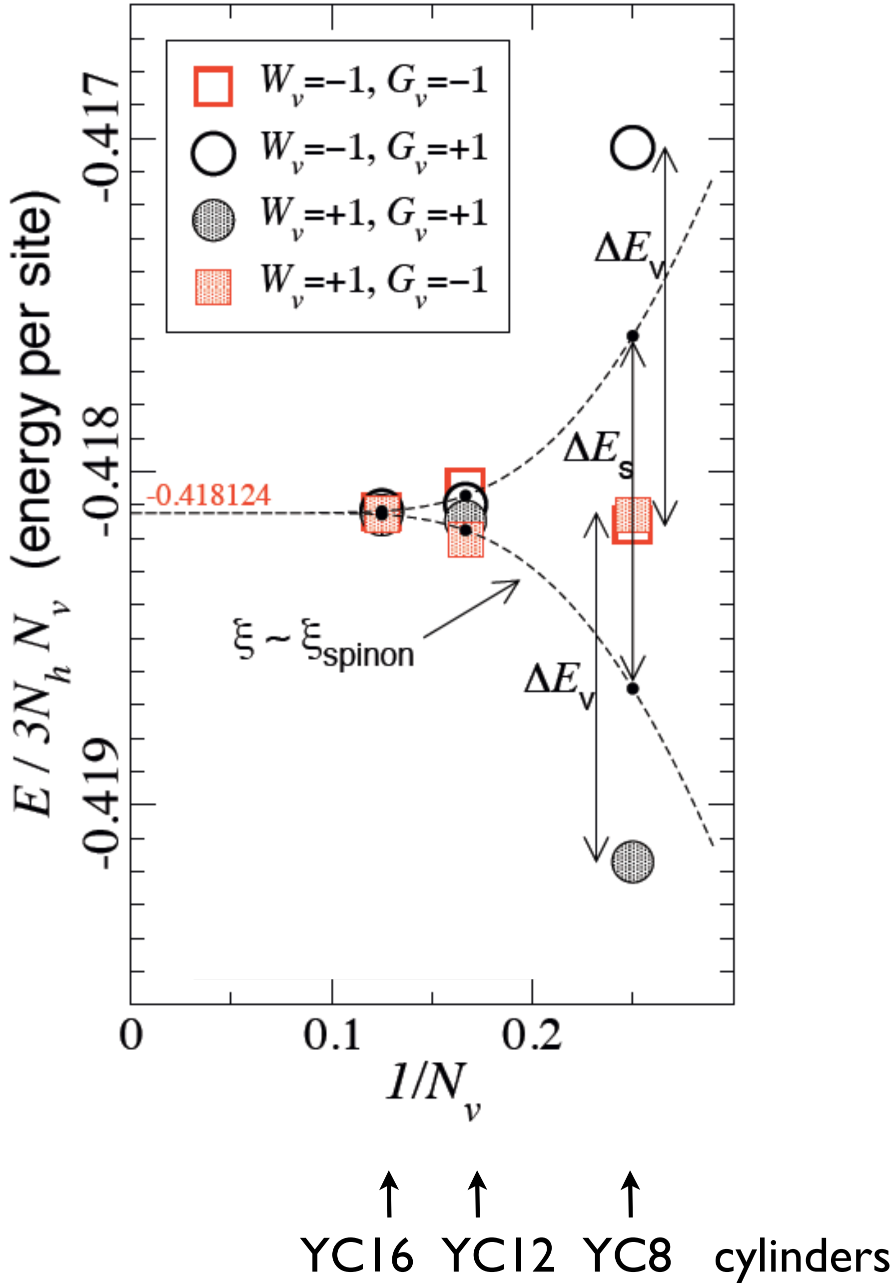
vison line = bond operator diag[1;1;-1] $= Z$

→ orthogonal in the limit of infinite cylinders $(N_h = \infty)$

Finite size scaling of RVB energy



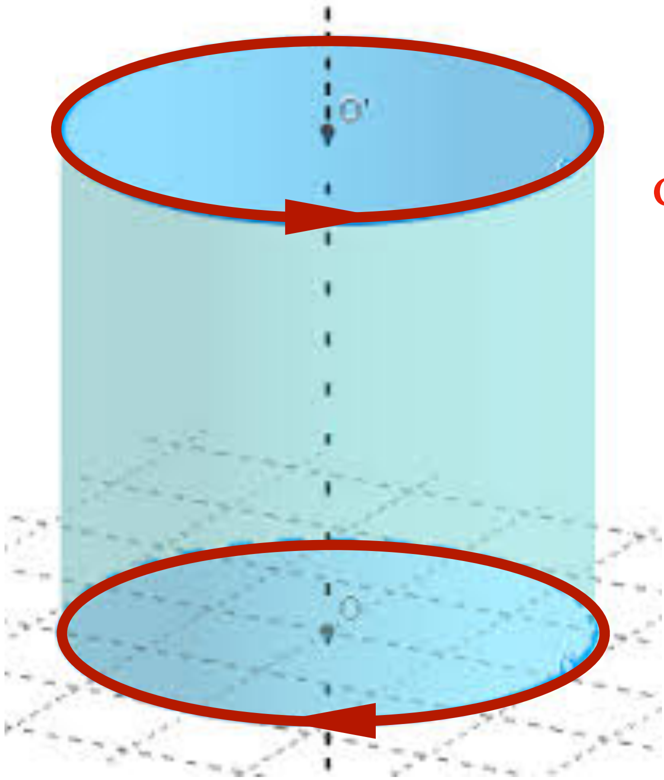
dynamical fluctuations of vison/spinon pairs



if T & P are broken :

chiral spin liquid

analogs of FQH states



Protected edge modes described by $SU(2)_k$ CFT

Can this be captured also by Tensor networks ?

Chiral topological spin liquids

- Topological (chiral) states are genuine in the field of the Fractional Quantum Hall effect

- Spin analogs on the lattice ?

Chiral SL (CSL) analogs of FQH states

1. Abelian CSL analog of Laughlin FQH state
2. Non-Abelian CSL analogs of non-Abelian FQH states (Moore-Read, Read-Rezayi, etc…)

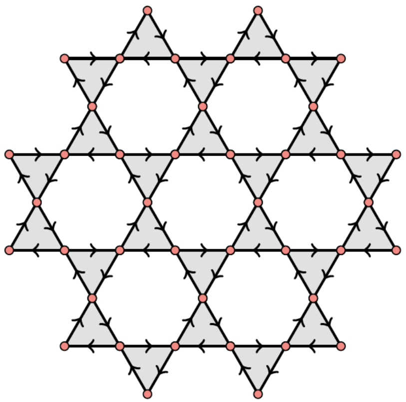
$\nu = \frac{1}{2}$ FQHS on a lattice (Kalmeyer-Laughlin, 1987):

(N=Nsites/2)

→ Paradigmatic "*Abelian* chiral spin liquid"

Chiral SL in microscopic spin-1/2 models ?

B. Bauer, L. Cincio, B. P. Keller, M. Dolfi, G. Vidal, S. Trebst, A. W. W. Ludwig
Nature Communications 5, 5137 (2014)



Mott phase of a large-U Hubbard model

$$H = J_{\mathrm{HB}} \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j + J_\chi \sum_{i,j,k \in \triangle} \vec{S}_i \cdot (\vec{S}_j \times \vec{S}_k)$$

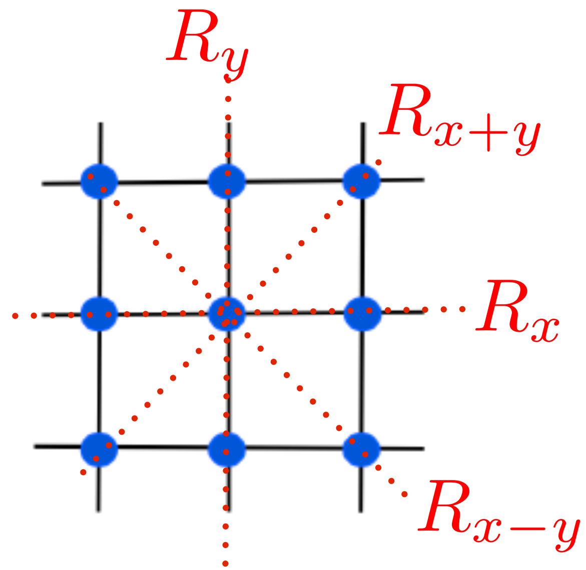
Edge modes : chiral c=1 CFT ($SU(2)_1$ WZW)

Simple requirement for a chiral PEPS

Necessary conditions:

$\forall \; G_i \in C_{4v}$ reflexion symmetries

$G_i|\Psi\rangle = |\Psi^*\rangle$ *Time-reversed partner*



Realized for a PEPS ansatz with the form:

$$\Psi = \Psi_s + i\,\Psi_g$$

Can be implemented from the symmetry of the local tensor A :

$$A = A_R + iA_I$$

$A_R \uparrow A_1$, $A_I \uparrow A_2$ irreps

Simple example : the chiral RVB PEPS

virtual states: $1/2 \oplus 0$ (D=3)

RVB tensor (NN singlets)

Generate NNN singlets (teleportation)

$$A = \lambda_1 R_1 + \lambda_2 R_2 + i\lambda_{\rm chiral} I$$

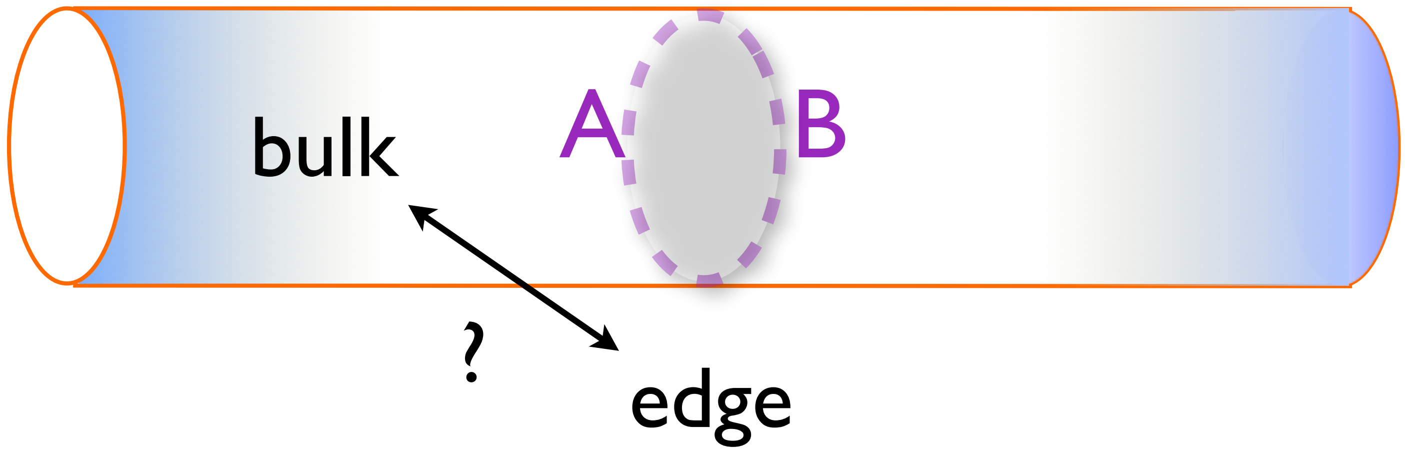
Point group sym.: s-wave ($\lambda_1 R_1 + \lambda_2 R_2$); g-wave ($I$)

inherits $\mathbb{Z}_2$ gauge symmetry

DP, J. Ignacio Cirac and Norbert Schuch, Phys. Rev. B **91**, 224431 (2015)

DP, Norbert Schuch and Ian Affleck, Phys. Rev. B **93**, 174414 (2016)

«Holographic» framework



$$\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$$

Reduced density matrix

Li & Haldane, 2008

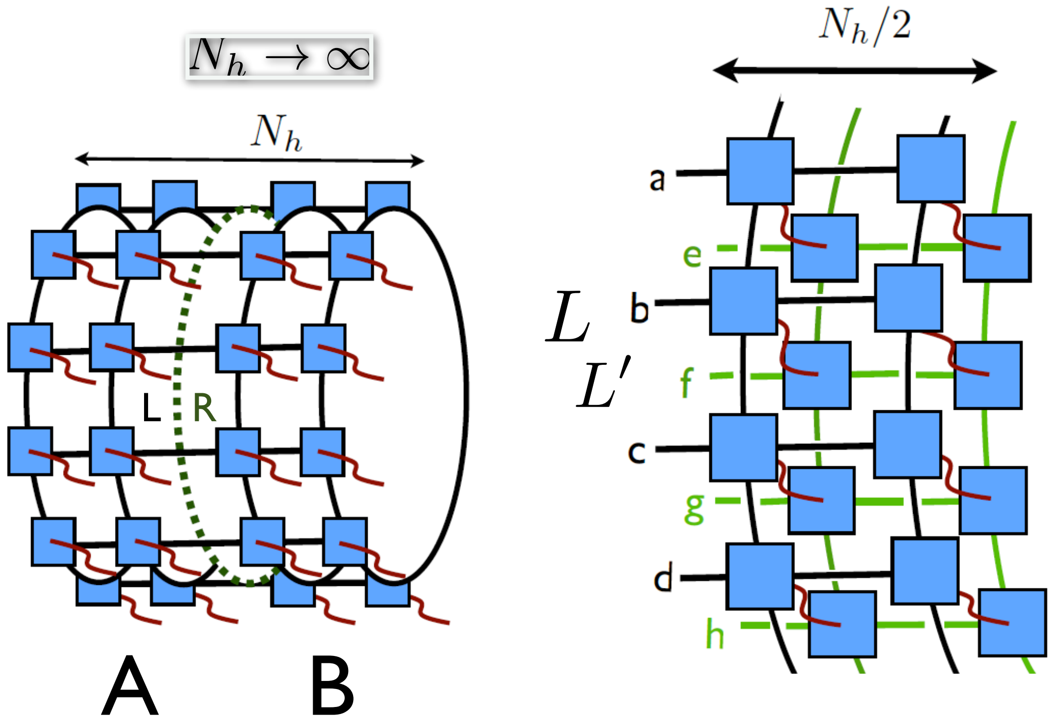
Regnault, Bernevig & Haldane, 2009

$$\rho_A = \exp(-H_b)$$

Entanglement Hamiltonian

"Haldane" Conjecture:

Precise correspondence between the **entanglement spectrum** of a FQH system partitioned into two sub-systems linked by some "edge" and the true sub-system spectrum



ρ_A

σ_b

lives” on the boundary

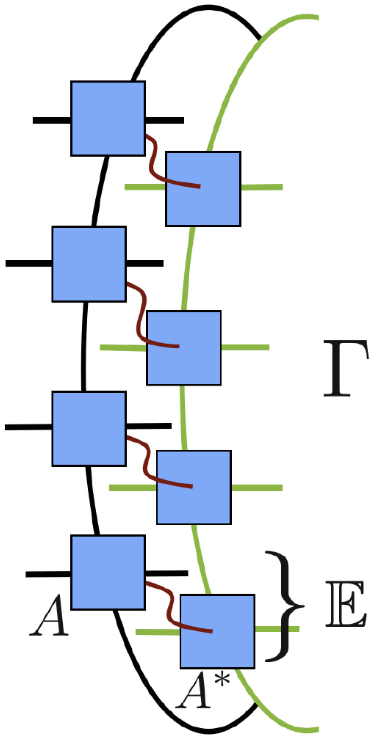
Basic formula: $\rho_A = U\sigma_b^2 U^\dagger$

isometry: maps 2D onto 1D

J. Ignacio Cirac, DP, Norbert Schuch, Frank Verstraete
Phys. Rev. B 83, 245134 (2011)

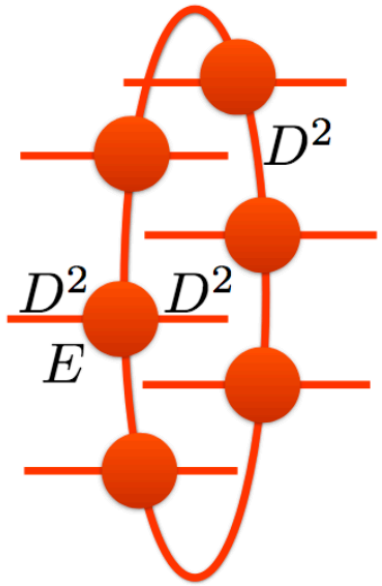
σ_b = leading eigenvector of TM

Transfer Matrix



$$D^{2N_v} \times D^{2N_v} \ \text{matrix}$$

=

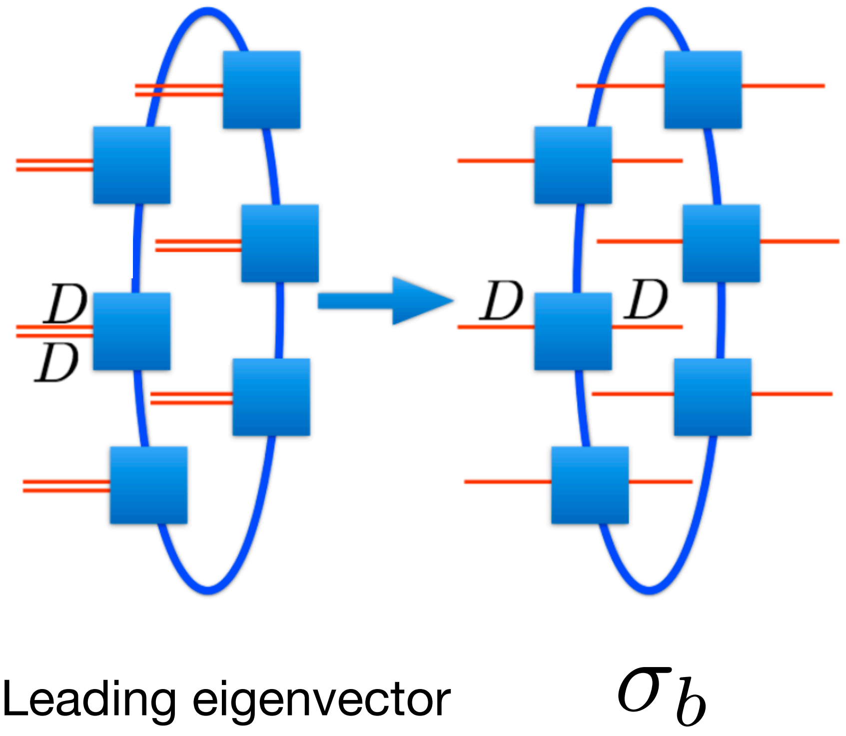


Build «double layer»
tensor network
by contracting physical variables

Leading eigenvector («fixed point») gives Entanglement Spectrum

Spectrum of TM provides correlation lengths

How to compute σ_b? -> leading eigenvector of transfer matrix



$$\sigma_b^2 = \exp(-H_b^{\mathrm{edge}})$$

Compare spectrum of H_b^{edge} to predictions of chiral CFT's

SU(2)$_1$ Entanglement spectrum

WZW SU(2)$_1$ CFT

$$E_{\rm CFT}(S_z, m_n, n) = \frac{\pi u}{N_v}\left(-\frac{c}{24} + S_z^2 + m_n n\right)$$

= Luttinger Liquid
at SU(2)-symmetric point

Even sector

Table 15.1. States in the lowest grades of the $\widehat{su}(2)_1$ module $L_{[1,0]}$.

L_0	S_z: −2	−1	0	1	2	$su(2)$ decomposition
0			1			(0)
1		1	1	1		(2)
2		1	2	1		(2)+(0)
3		2	3	2		2(2)+(0)
4	1	3	5	3	1	(4)+2(2)+2(0)
5	1	5	7	5	1	(4)+4(2)+2(0)
6	2	7	11	7	2	2(4)+5(2)+4(0)

Odd sector

Table 15.2. States in the lowest grades of the $\widehat{su}(2)_1$ module $L_{[0,1]}$.

L_0	$2S_z$: −2	−1	0	1	2	3	$su(2)$ decomposition
$\frac{1}{4}$			1	1			(1)
$\frac{5}{4}$			1	1			(1)
$\frac{9}{4}$		1	2	2	1		(3)+(1)
$\frac{13}{4}$		1	3	3	1		(3)+2(1)
$\frac{17}{4}$		2	5	5	2		2(3)+3(1)
$\frac{21}{4}$		3	7	7	3		3(3)+4(1)
$\frac{25}{4}$	1	5	11	11	5	1	(5)+4(3)+6(1)

Philippe Di Francesco
Pierre Mathieu
David Sénéchal

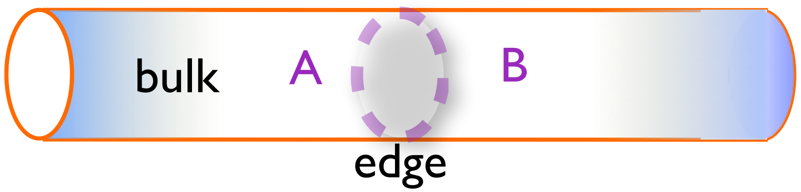
Conformal Field Theory

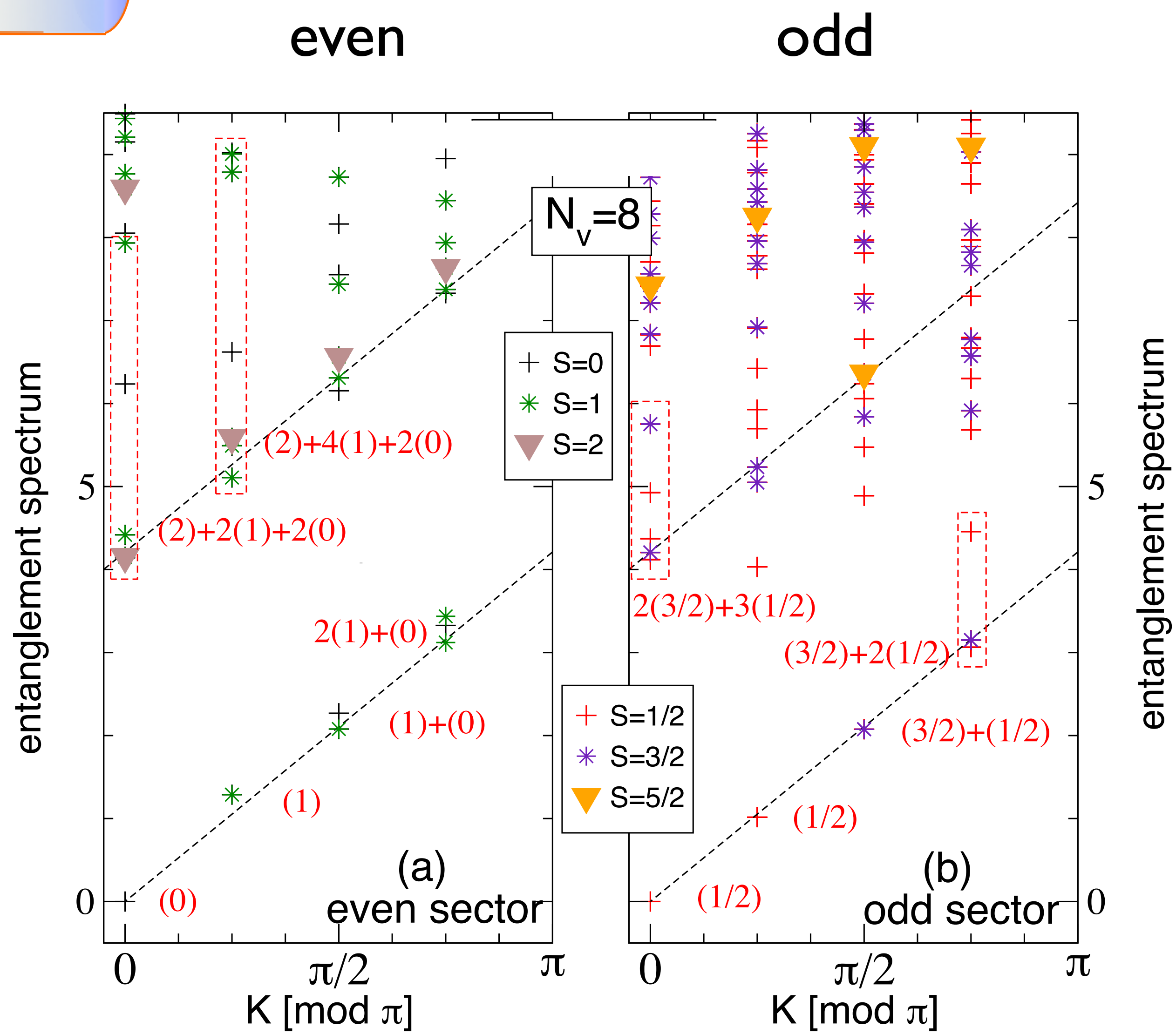
$$E \equiv N_v e_0 + e_{\rm topo} + E_{\rm CFT}$$

$$e_0 = \frac{\pi c}{6u}$$

$$e_{\rm topo} = -\ln 2/2$$

Entanglement spectrum





Entanglement entropy

$$S_n = -\frac{1}{n-1}\mathrm{Ln}\{\mathrm{Tr}(\rho_\mathrm{A})^\mathrm{n}\}$$

(Renyi)

$$S_\mathrm{VN} = -\mathrm{Tr}\{\rho_A \ln \rho_A\}$$

(Von Neumann)

"area" law

$$S_\mathrm{VN} \sim C N_v - S_\mathrm{TE}$$

Kitaev & Preskill, 2006

Levin & Wen, 2006

subleading correction to area law:
topological entropy

EE of chiral liquid (CFT predictions)

Kitaev & Preskill, 2006

Levin & Wen, 2006

Renyi: $$S_q \sim \frac{q+1}{q} e_0 N_v + e_{\rm topo}$$

$$e_{\rm topo} = -\ln 2/2$$

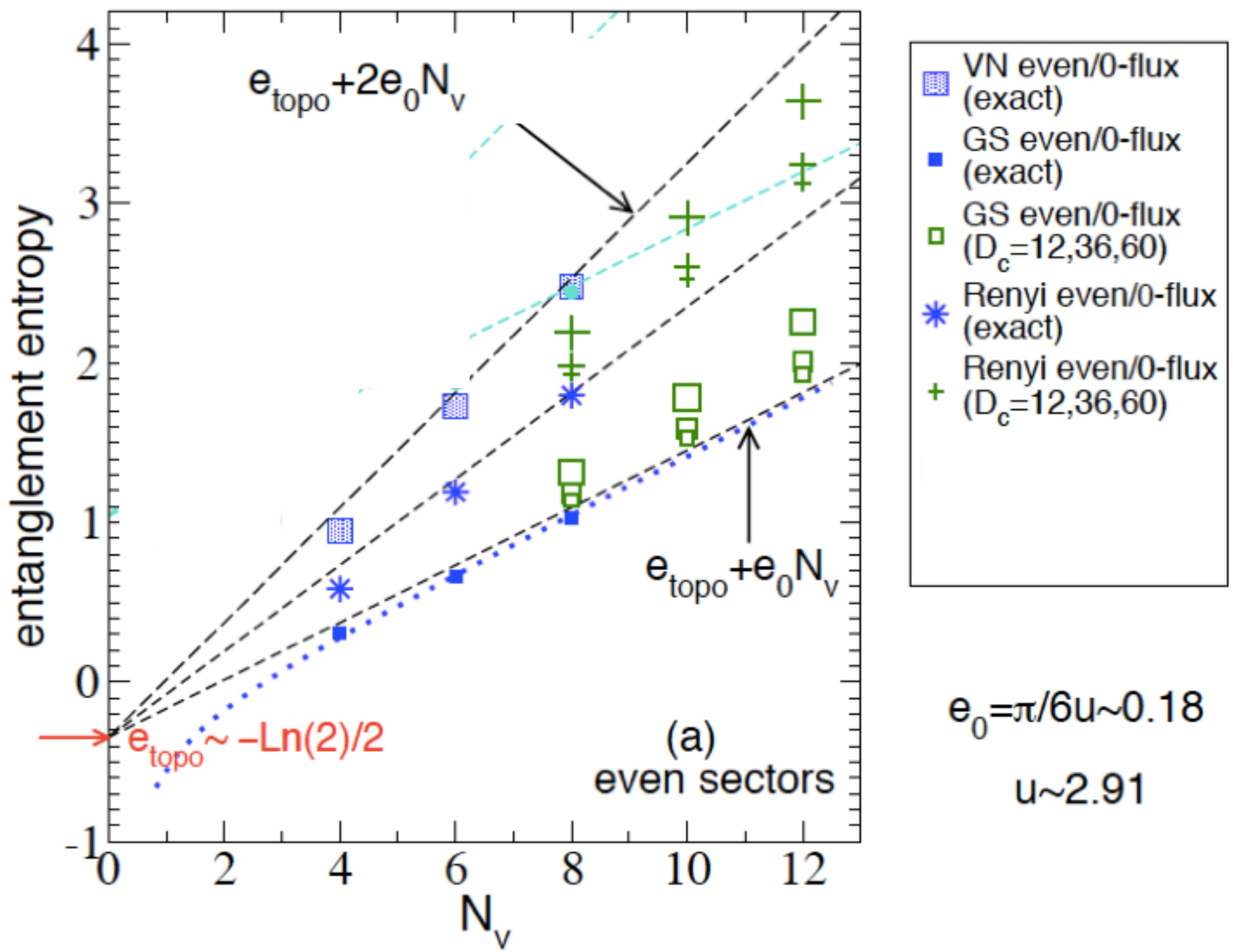
Von Neumann $$S_{VN} \sim (2e_0) N_v + e_{\rm topo}$$

$$S_\infty \sim e_0 N_v + e_{\rm topo} + \frac{\pi u}{N_v} \alpha^{(p)}$$

$$\alpha^{(e)} \sim -\frac{c}{24}$$

$$\alpha^{(o)} \sim \frac{1}{4} - \frac{c}{24}$$

EE of square chiral RVB liquid (PEPS)



$e_0=\pi/6u\sim0.18$

$u\sim2.91$

Parent hamiltonian of non-Abelian (bosonic) Moore-Read CSL (I)

I. Glasser, I. Cirac, G. Sierra & A. Nielsen (2015)

Bosonic Moore-Read Pfaffian state at $\nu = 1\,(q = 1)$

$$\psi(w_1, \ldots, w_M) \propto \prod_{i<j} (w_i - w_j)^q \mathrm{Pf}\left[\frac{1}{\mathrm{w}_i - \mathrm{w}_j}\right] e^{-\frac{1}{4}\sum_i |w_i|^2}$$

non-Abelian anyons : $\sigma \times \sigma = 1 + \Psi$

like Kitaev's honeycomb non-Abelian phase

Can be written as CFT correlator of primary fields of $SU(2)_2$ CFT

→ Spin-1 singlet wave function

Parent hamiltonian of non-Abelian Moore-Read CSL (II)

Parent Hamiltonian is long range : truncation needed !

$$H = J_1 \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + J_2 \sum_{\langle\langle k,l \rangle\rangle} \mathbf{S}_k \cdot \mathbf{S}_l$$

$$+K_1 \sum_{\langle i,j \rangle} (\mathbf{S}_i \cdot \mathbf{S}_j)^2 + K_2 \sum_{\langle\langle k,l \rangle\rangle} (\mathbf{S}_i \cdot \mathbf{S}_j)^2$$

Spin-1 chiral HAFM

$$+K_c \sum_{\square} [\mathbf{S}_i \cdot (\mathbf{S}_j \times \mathbf{S}_k) + \mathbf{S}_j \cdot (\mathbf{S}_k \times \mathbf{S}_m)$$

$$+\mathbf{S}_i \cdot (\mathbf{S}_j \times \mathbf{S}_m) + \mathbf{S}_i \cdot (\mathbf{S}_k \times \mathbf{S}_m)]$$

optimize overlap $\langle \Psi_{\mathrm{GS}} | \Psi_{\mathrm{MR}} \rangle$

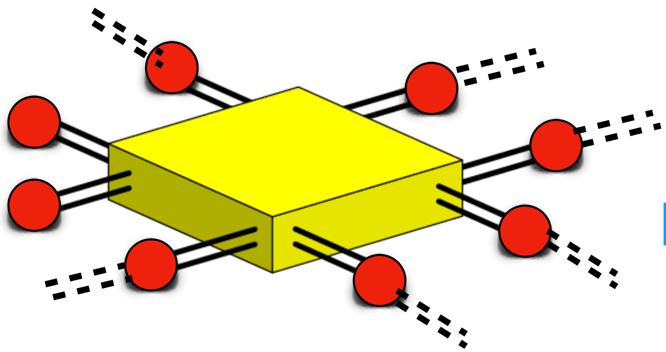
$J_1 = 1 \quad J_2 = 0.623 \quad K_1 = -0.176 \quad K_2 = 0.323 \quad K_c = 0.464$

iPEPS method (I)
CTMRG

- Environment constructed by renormalization of the corner transfer matrix (CTM)

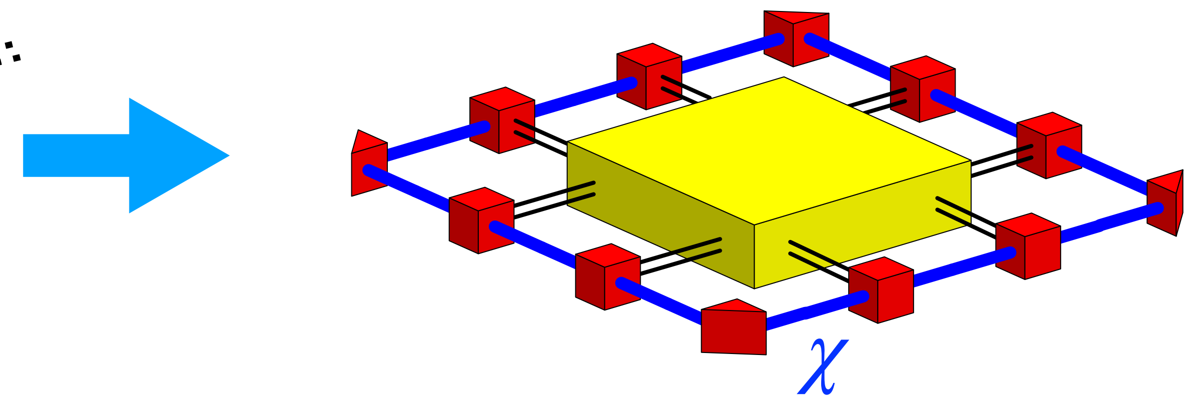
T. Nishino & K. Okunichi, J. Phys. Soc. J. **65**, 891 (1996)
R. Orus & G. Vidal, Phys. Rev. B **80**, 094403 (2009)

Infinite lattice



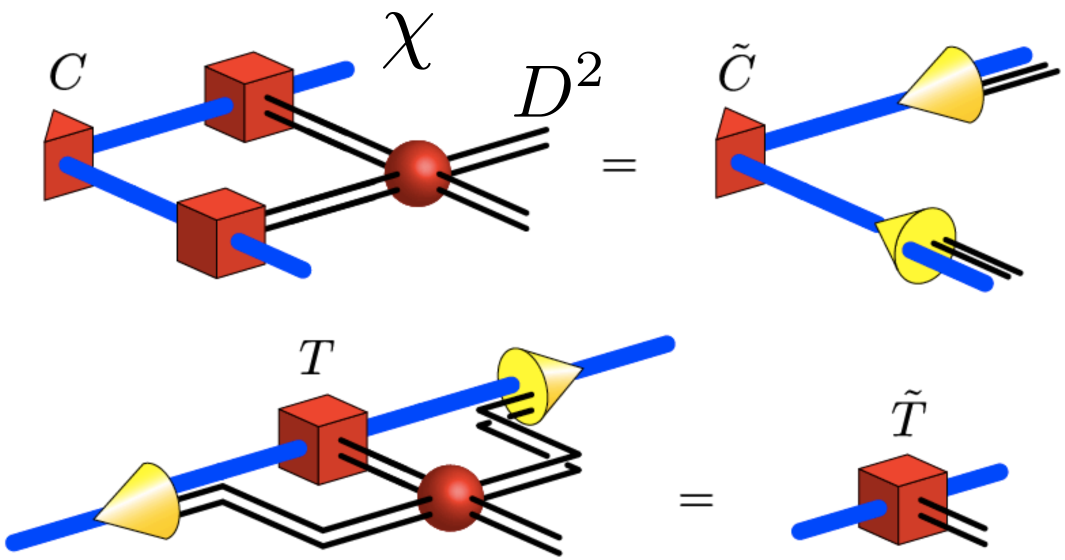
$$\langle \Psi_{\mathrm{PEPS}} | H_{\square} | \Psi_{\mathrm{PEPS}} \rangle$$

$=$



CTM Renormalization Group algorithm

Isometry $\chi \times \chi D^2$

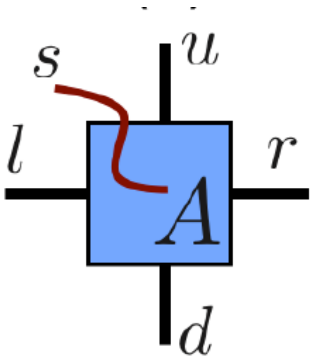


SVD decomposition :
keep χ largest weights

Renormalize the edge tensor

General construction using a classification of SU(2)-invariant PEPS

M. Mambrini, R. Orus & DP, Phys. Rev. B 94, 205124 (2016)



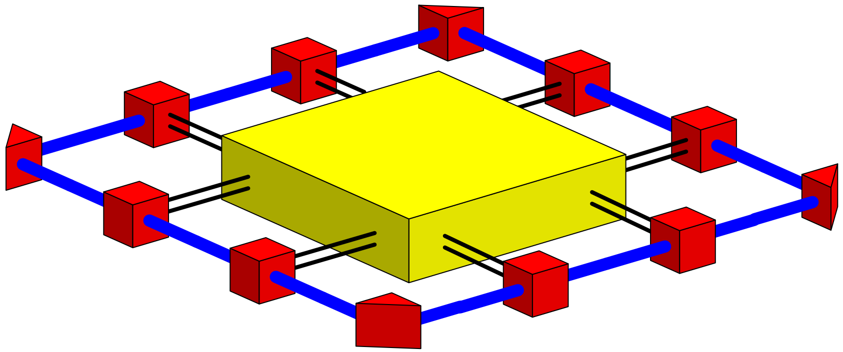
* virtual space
* Irreps of point group
(C4v for square lattice)

Chiral PEPS ansatz: $A = A_R + iA_I$

$$A_R = \sum_\alpha \lambda_\alpha A_\alpha^{(A_1)} \quad A_I = \sum_\beta \gamma_\beta A_\beta^{(A_2)}$$

iPEPS method (II)
Variational optimization

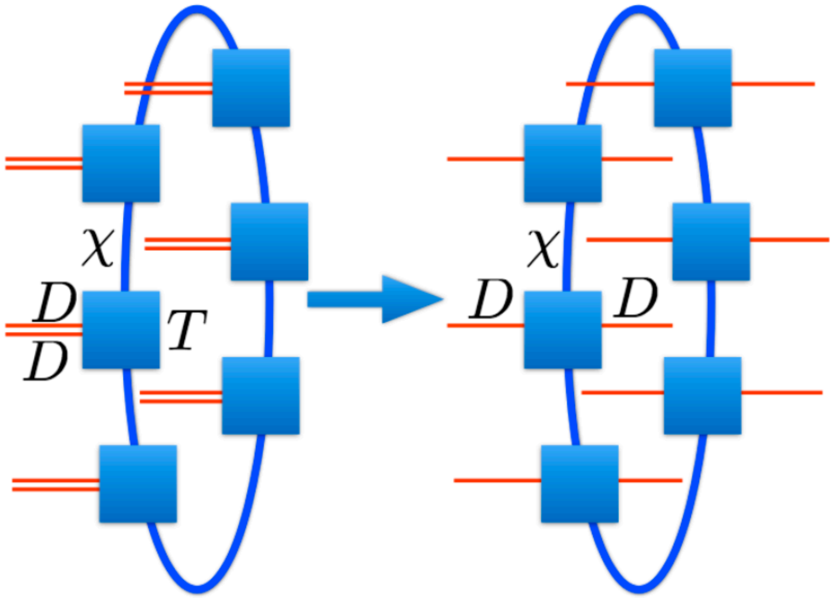
$$E\{\lambda_\alpha, \gamma_\beta\} =$$



- Variational optimisation scheme based on a conjugate gradient method

L. Vanderstraeten, J. Haegeman, P. Corboz, F. Verstraete, Phys. Rev. B **94**, 155123 (2016)
DP & M. Mambrini, Phys. Rev. B **96**, 014414 (2017)
DP, Phys. Rev. B **96**, 121118 (2017)

How to compute σ_b? -> leading eigenvector of transfer matrix constructed directly from the T environment tensor



Leading eigenvector from a ring of T tensors

σ_b

$$\sigma_b^2 = \exp(-H_b^{\text{edge}})$$

Compare spectrum of H_b^{edge} to predictions of chiral CFT's

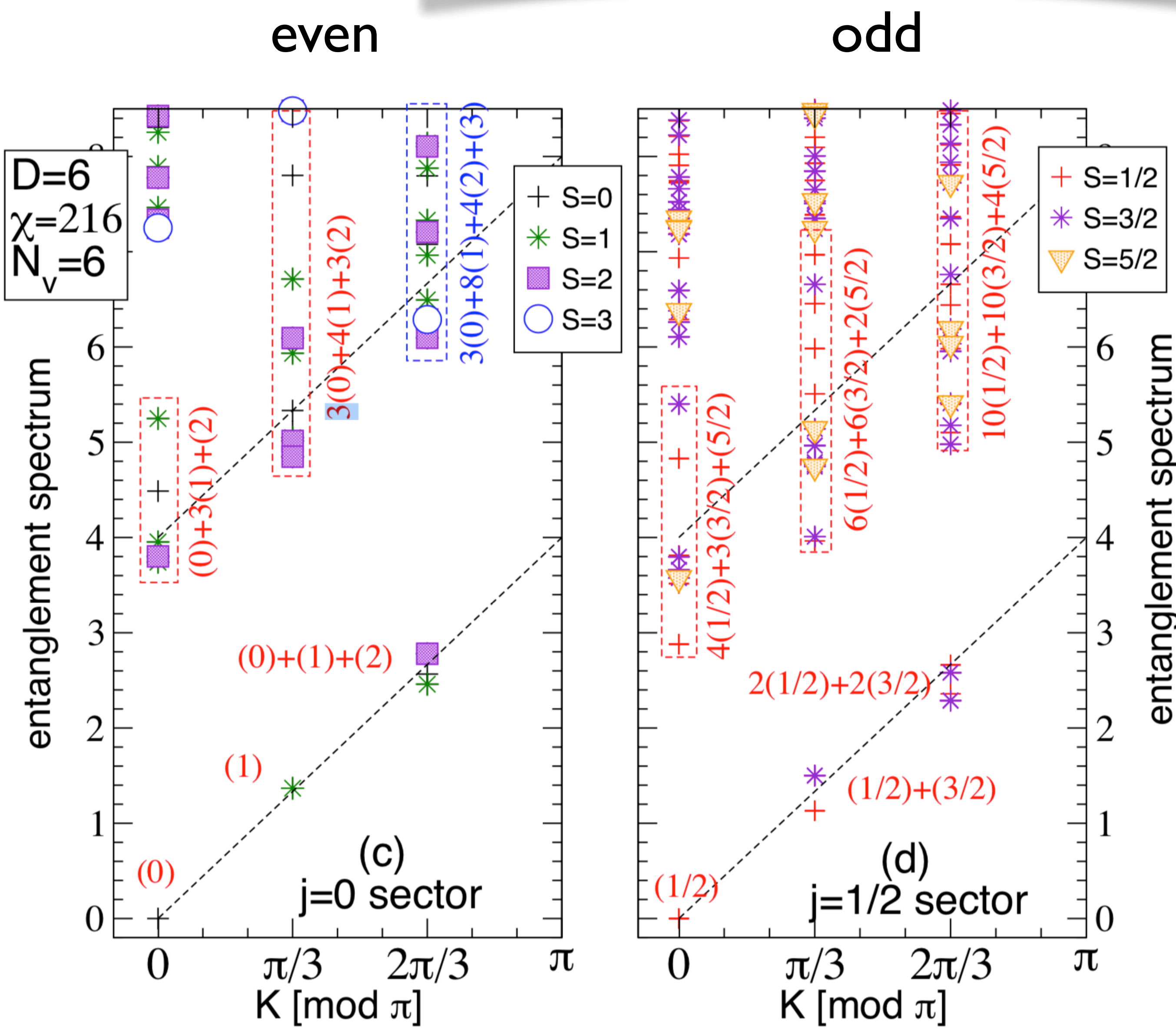
Conformal tower of $SU(2)_2$ CFT

Central charge c=3/2
Majorana (Ising) + boson

Conformal tower:

$n\backslash j$	0	$\frac{1}{2}$	1
0	(0)	$(\frac{1}{2})$	(1)
1	(1)	$(\frac{1}{2})+(\frac{3}{2})$	$(0)+(1)$
2	$(0)+(1)+(2)$	$2(\frac{1}{2})+2(\frac{3}{2})$	$(0)+2(1)+(2)$
3	$(0)+3(1)+(2)$	$4(\frac{1}{2})+3(\frac{3}{2})+(\frac{5}{2})$	$2(0)+3(1)+2(2)$
4	$3(0)+4(1)+3(2)$	$6(\frac{1}{2})+6(\frac{3}{2})+2(\frac{5}{2})$	-
5	$3(0)+8(1)+4(2)+(3)$	$10(\frac{1}{2})+10(\frac{3}{2})+4(\frac{5}{2})$	-

Entanglement spectrum for spin-1 CSL



"tower of states" of chiral $SU(2)_2$ CFT & c~1.5

non-Abelian Moore-Read Chiral SL

Conclusions

- Tensor network techniques in general, PEPS in particular, are promising conceptual framework and numerical tool for correlated systems in 2D
- PEPS can be generalized to fermions, SU(N) symetries, etc…
- Ideal framework to investigate systems with topological order

Many contributors including…

PEPS spin-1/2 chiral spin liquids

DP, J. Ignacio Cirac and Norbert Schuch,
Phys. Rev. B 91, 224431 (2015)

DP, Norbert Schuch and Ian Affleck,
Phys. Rev. B 93, 174414 (2016)
(Editor's suggestion)

Classification of SU(2) spin liquids

Matthieu Mambrini, Roman Orus & DP,
Phys. Rev. B 94, 205124 (2016)

Chiral SL in frustrated models with iPEPS

DP, Phys. Rev. B 96, 121118 (2017)

Ji-Yao Chen, L. Vanderstraeten, S. Capponi & DP,
Phys. Rev. B 98, 184409 (2018)