Field theory of composite fermions

Dam Thanh Son (University of Chicago)
Les Houches, 09/06/2019

Part 1: FQHE and field theoretic duality, Dirac composite fermion

Part 2: Hydrodynamics of the composite fermions, "emergent graviton"

Fractional quantum Hall effect and field-theoretic dualities

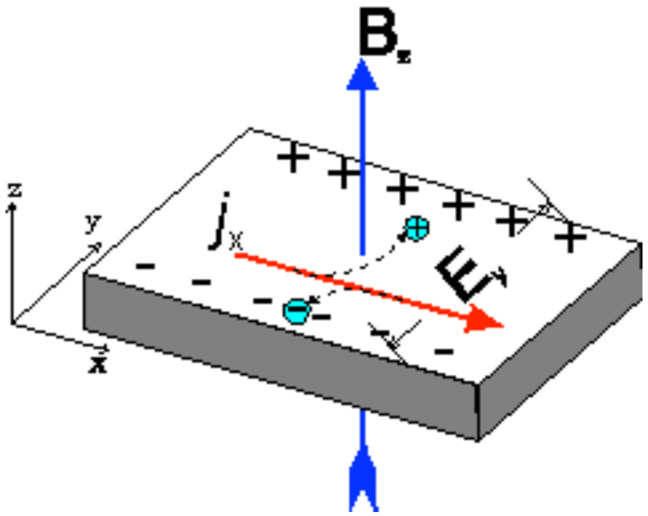
Plan

- Fractional quantum Hall effect
- Composite fermion
- Duality

The quantum Hall effect

The microscopic theory of the quantum Hall effect

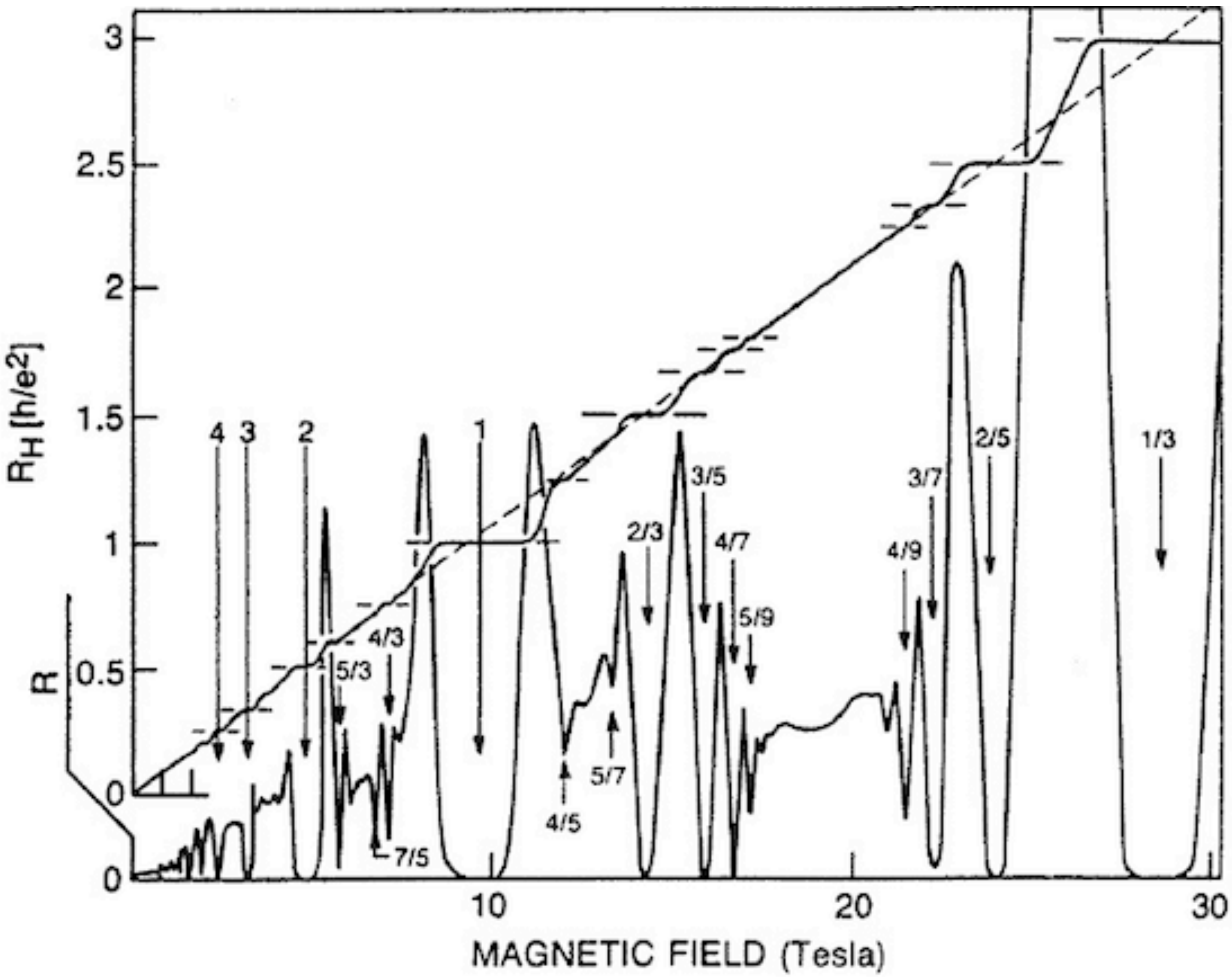
2D electrons in a magnetic field



At the microscopic level: a known Hamiltonian

$$H = \sum_a \frac{(\mathbf{p}_a + e\mathbf{A}_a)^2}{2m} + \sum_{\langle a,b \rangle} \frac{e^2}{|\mathbf{x}_a - \mathbf{x}_b|}$$

+ impurities



R_H [h/e^2]
R
MAGNETIC FIELD (Tesla)
0
0.5
1
1.5
2
2.5
3
0
10
20
30
4
3
2
1
5/3
4/3
7/5
4/5
5/7
2/3
3/5
4/7
5/9
4/9
3/7
2/5
1/3

Landau levels and IQHE

- Ignore interactions between electrons
- Energy levels of charged particle in magnetic field in 2D: Landau levels
- When some Landau level fully occupied: integer quantum Hall effect (IQHE) von Klitzing et al 1980

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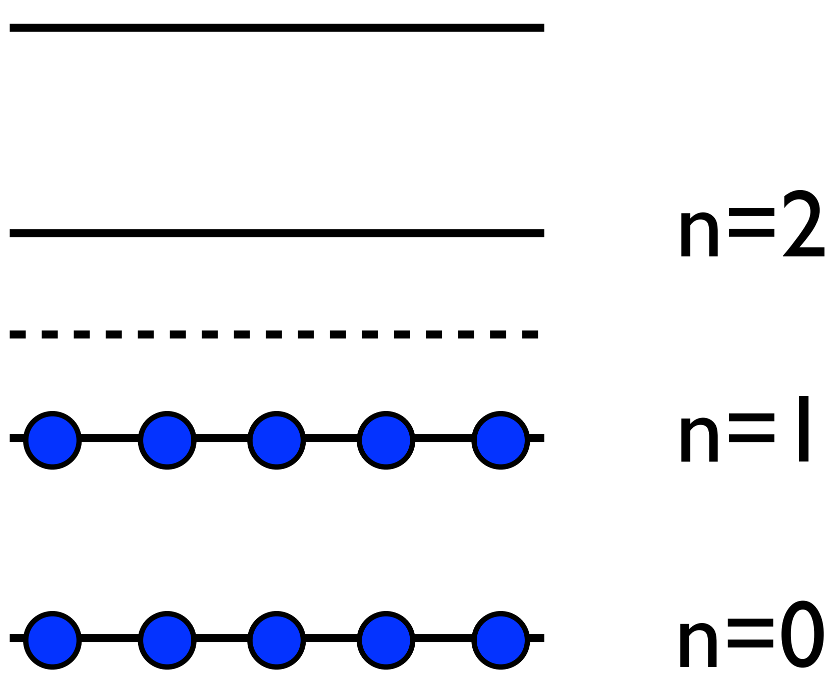
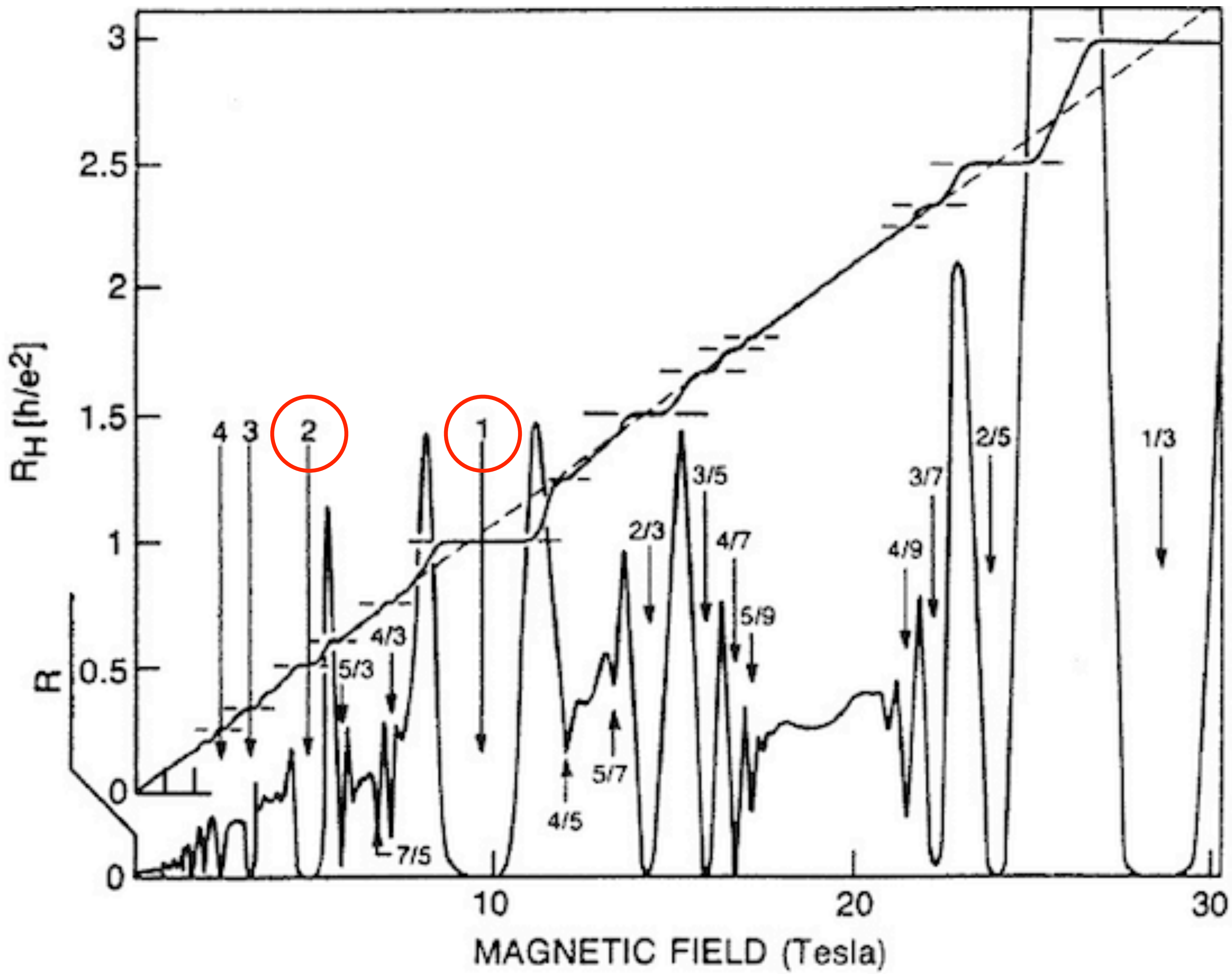


$$\Delta = \frac{B}{m}$$

degeneracy $\dfrac{BA}{2\pi}$



R_H [h/e²]
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3
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5/9
4/9
3/7
2/5
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10
20
30
MAGNETIC FIELD (Tesla)

The fractional quantum Hall effect (FQHE)

n=2

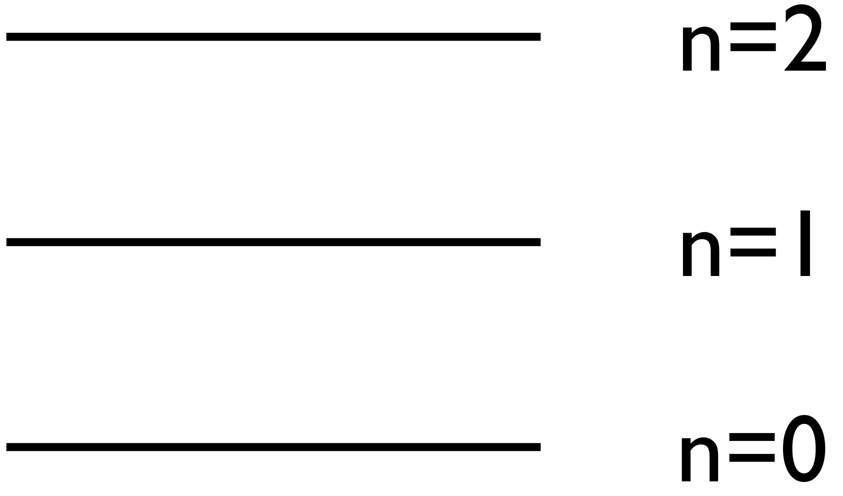
n=1

n=0

The fractional quantum Hall effect (FQHE)

filling factor

$$\nu = \frac{\text{number of electrons}}{\text{degeneracy of a LL}}$$



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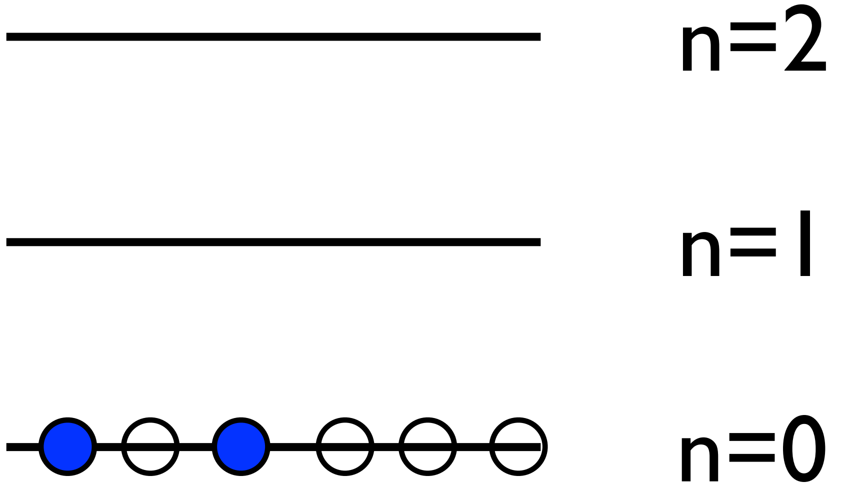

$\nu < 1$

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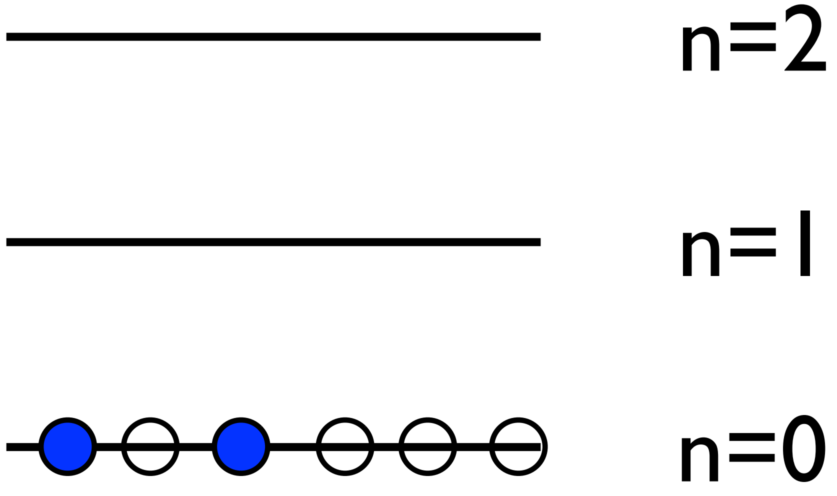

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- without interaction: large ground-state degeneracy

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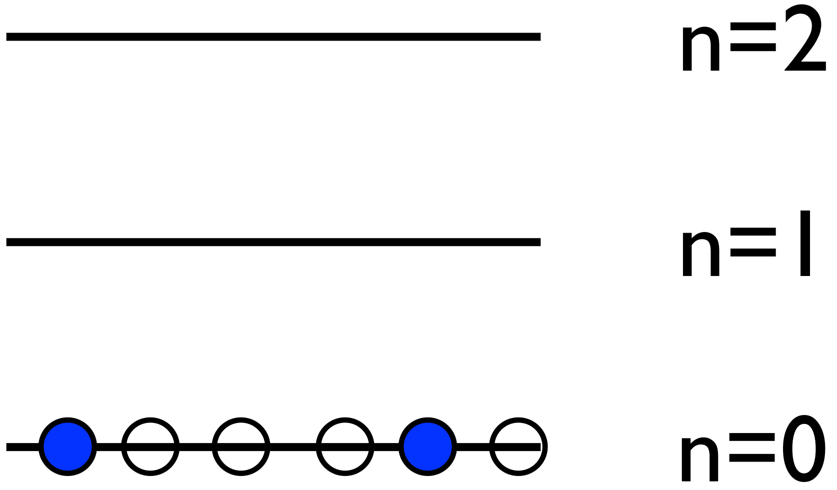

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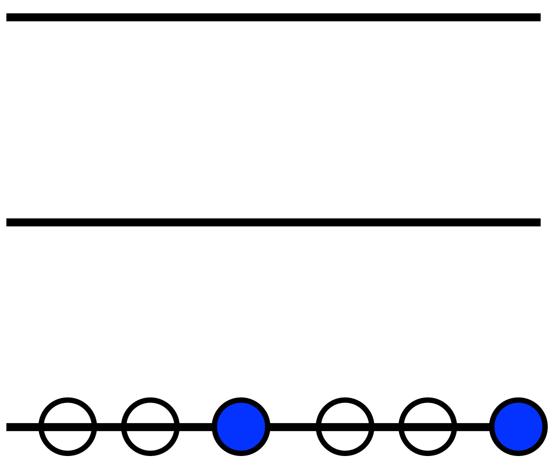

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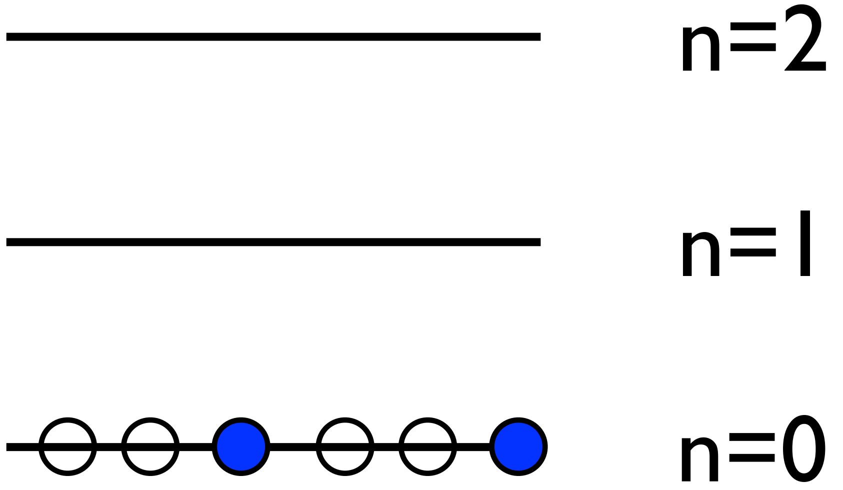
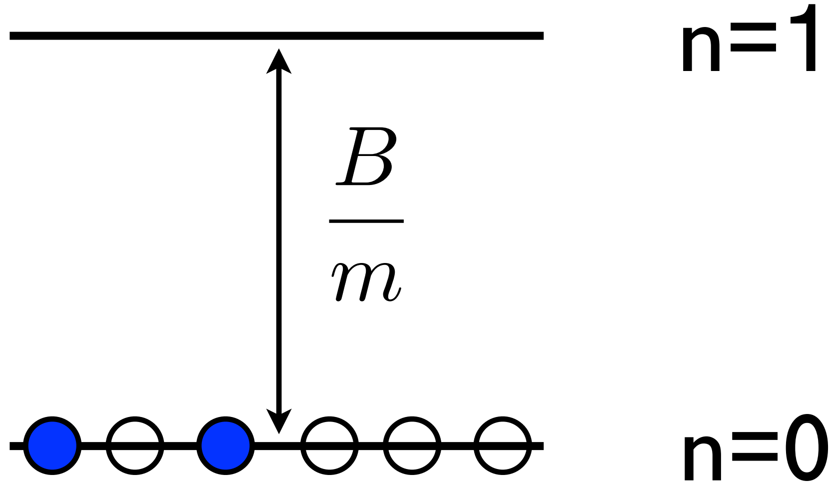


$\nu < 1$

- without interaction: large ground-state degeneracy
- Interactions are essential for determining the ground state
- gapped at nu=1/3, ungapped at nu=1/2

Lowest Landau level limit

$$H = \sum_a \frac{(\mathbf{p}_a + e\mathbf{A}_a)^2}{2m} + \sum_{\langle a,b \rangle} \frac{e^2}{|\mathbf{x}_a - \mathbf{x}_b|}$$



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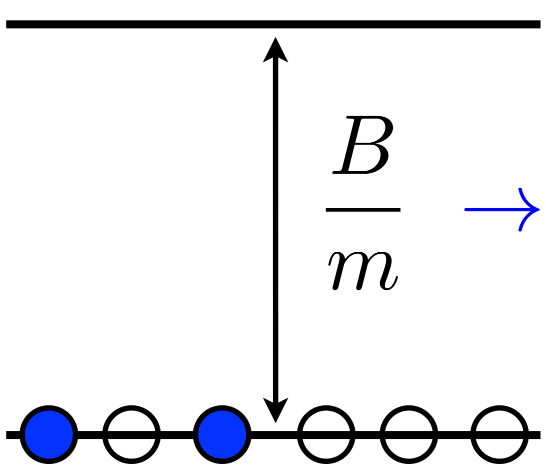
$m \to 0$



n=1

$\frac{B}{m} \to \infty$

n=0

Lowest Landau level limit

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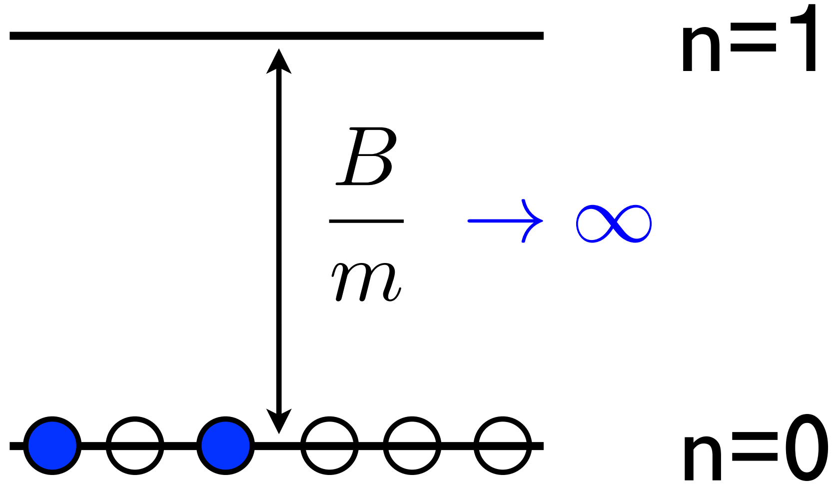
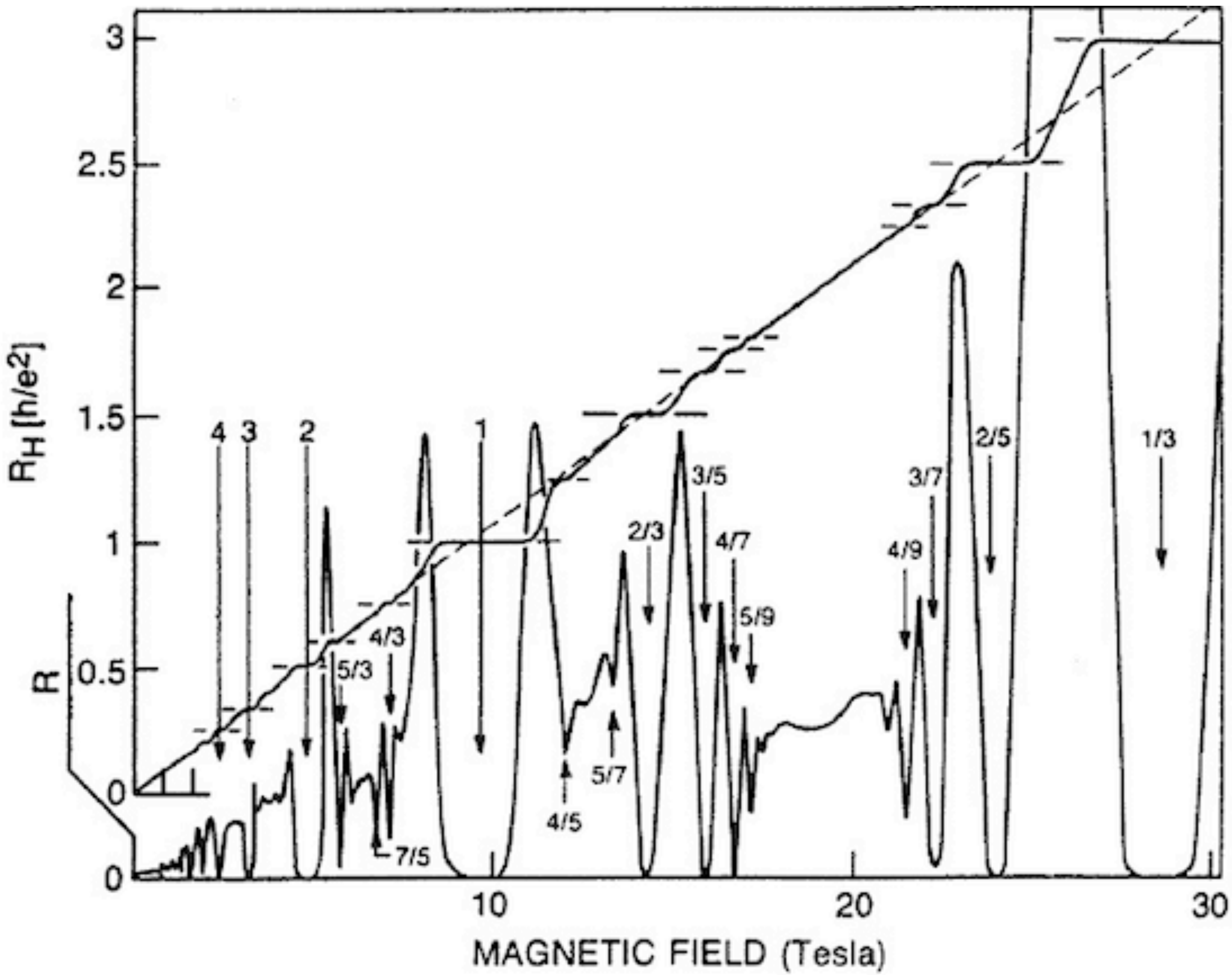


$$H = P_{\mathrm{LLL}} \sum_{a,b} \frac{e^2}{|\mathbf{x}_a - \mathbf{x}_b|}$$

Projection to
lowest Landau level



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4/5
5/7
10
20
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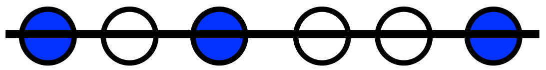
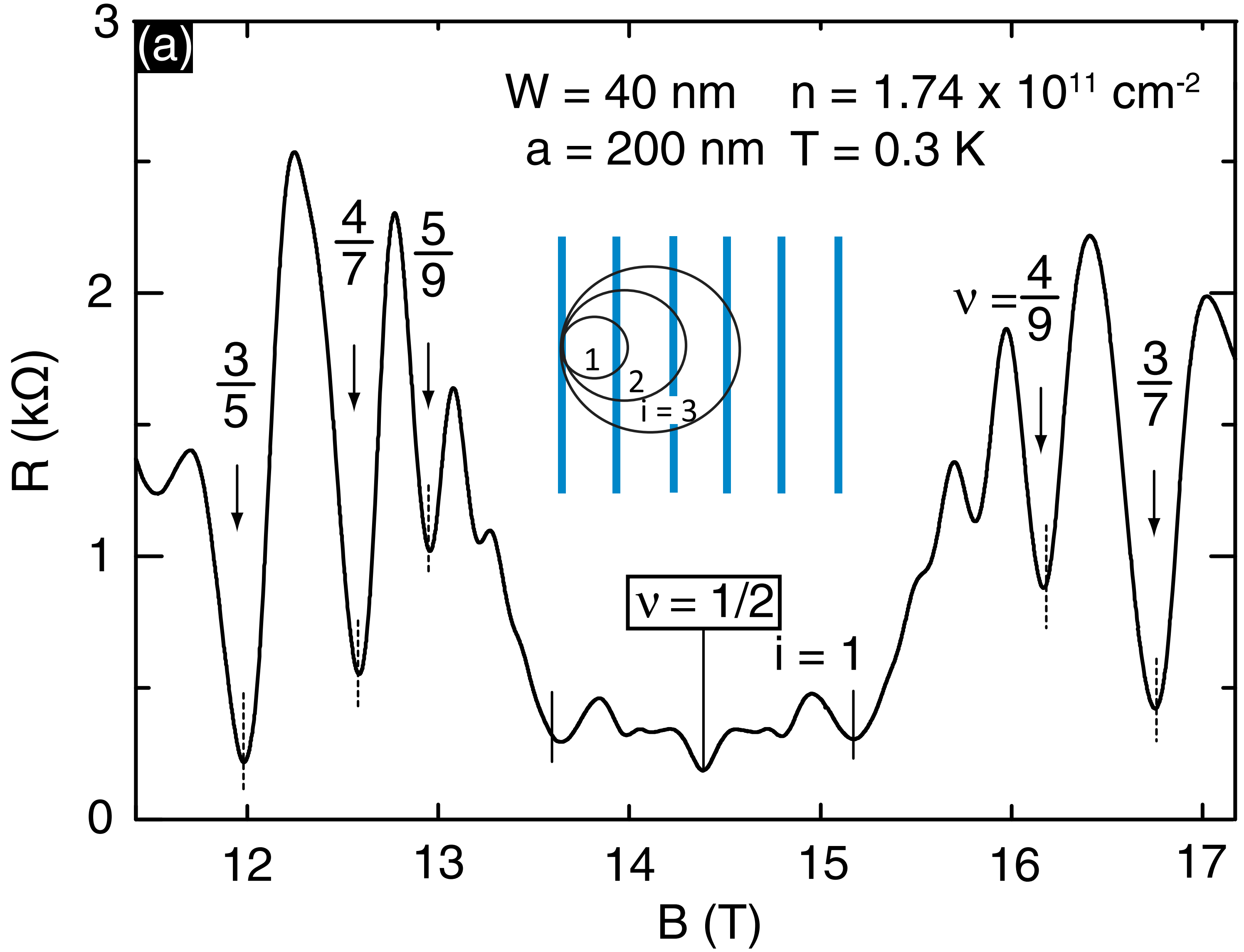
MAGNETIC FIELD (Tesla)

- First successful theory: Laughlin's wave function
- Can explain $\nu=1/3$ and $\nu=1/5$ plateaus
- but to explain the other plateaus one needs a new idea
  - A quasiparticle: composite fermion (CF)

When ν approaches 1/2, a quasiparticle appears which moves practically in straight line





(Kamburov et al, 2014)

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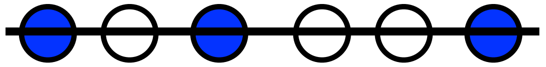
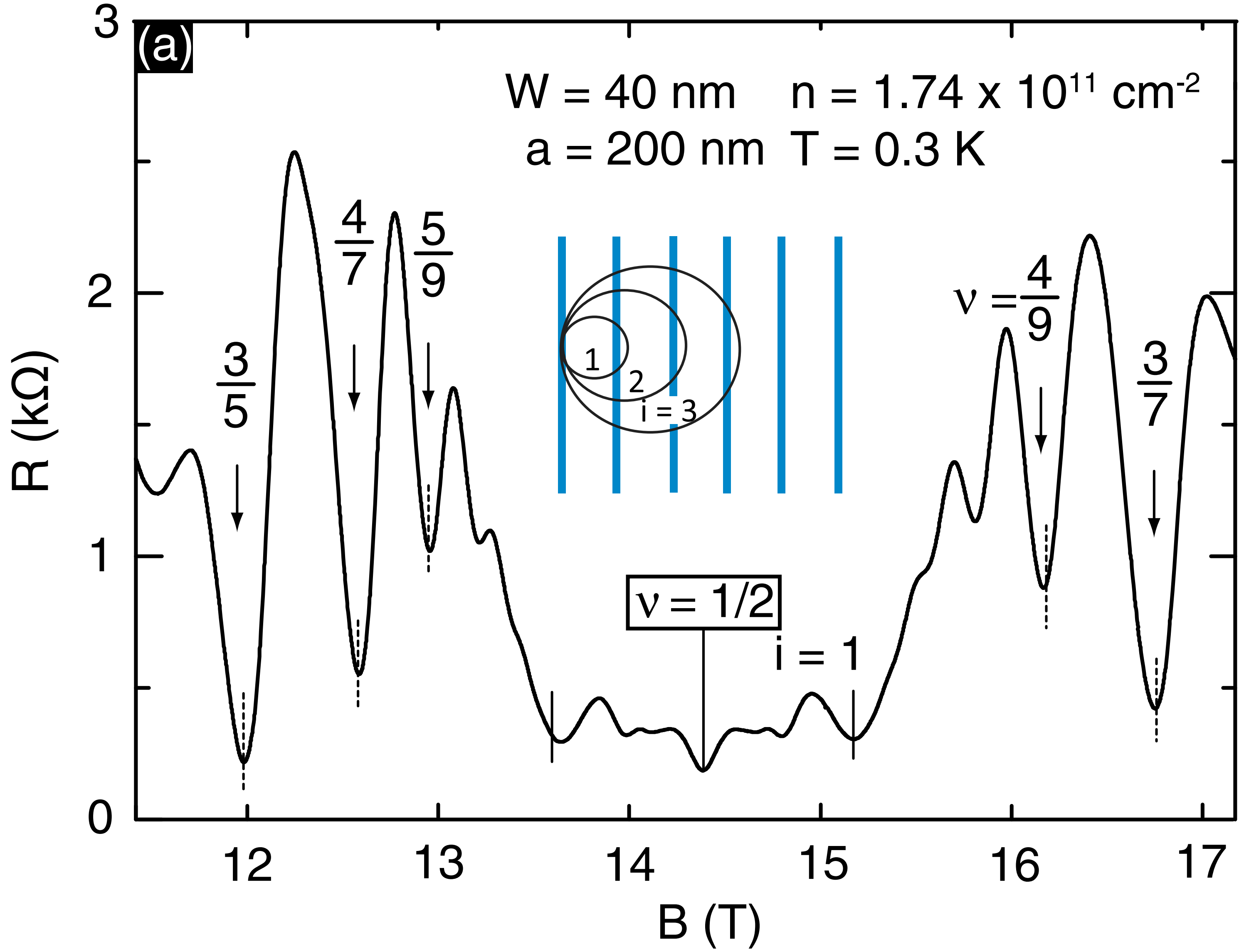




(Kamburov et al, 2014)

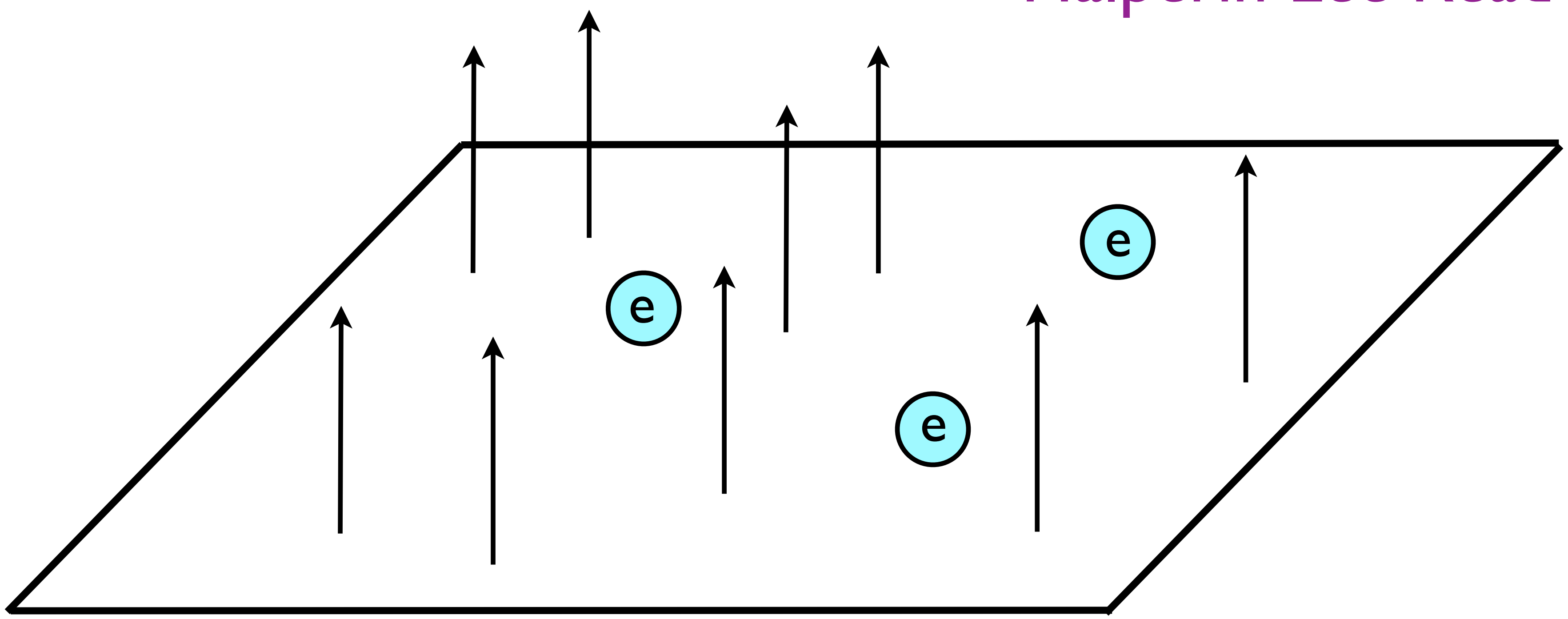
What is the nature of this quasiparticle?

Composite fermion

- The standard picture: the quasiparticle is a “composite fermion” = electron + 2 flux quanta

Flux attachment

Jain, Lopez Fradkin, Ovchinnikov,
Halperin Lee Read ~ 1990



$\nu = \frac{1}{3}$ $\uparrow\uparrow\uparrow$ per e

Flux attachment

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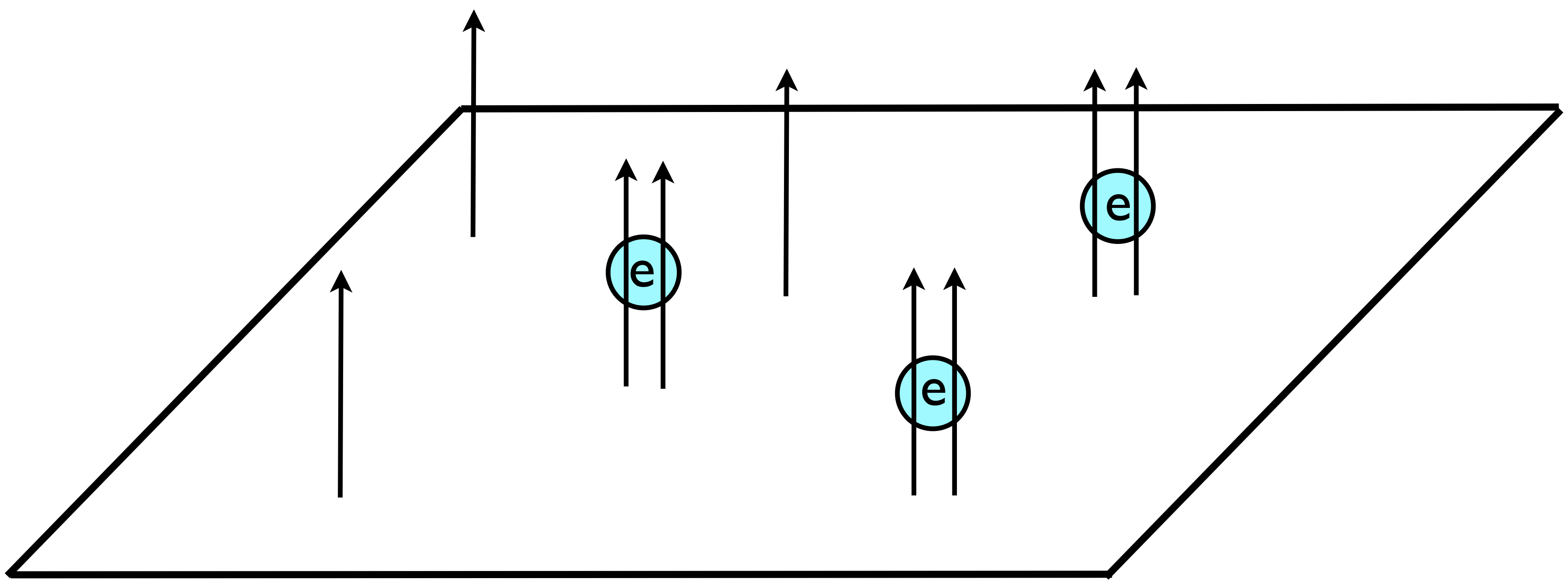

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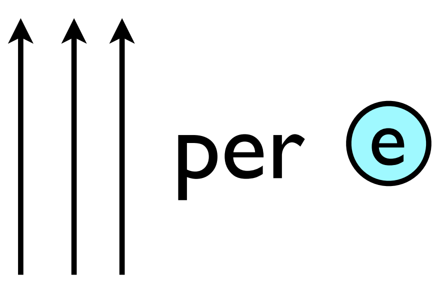

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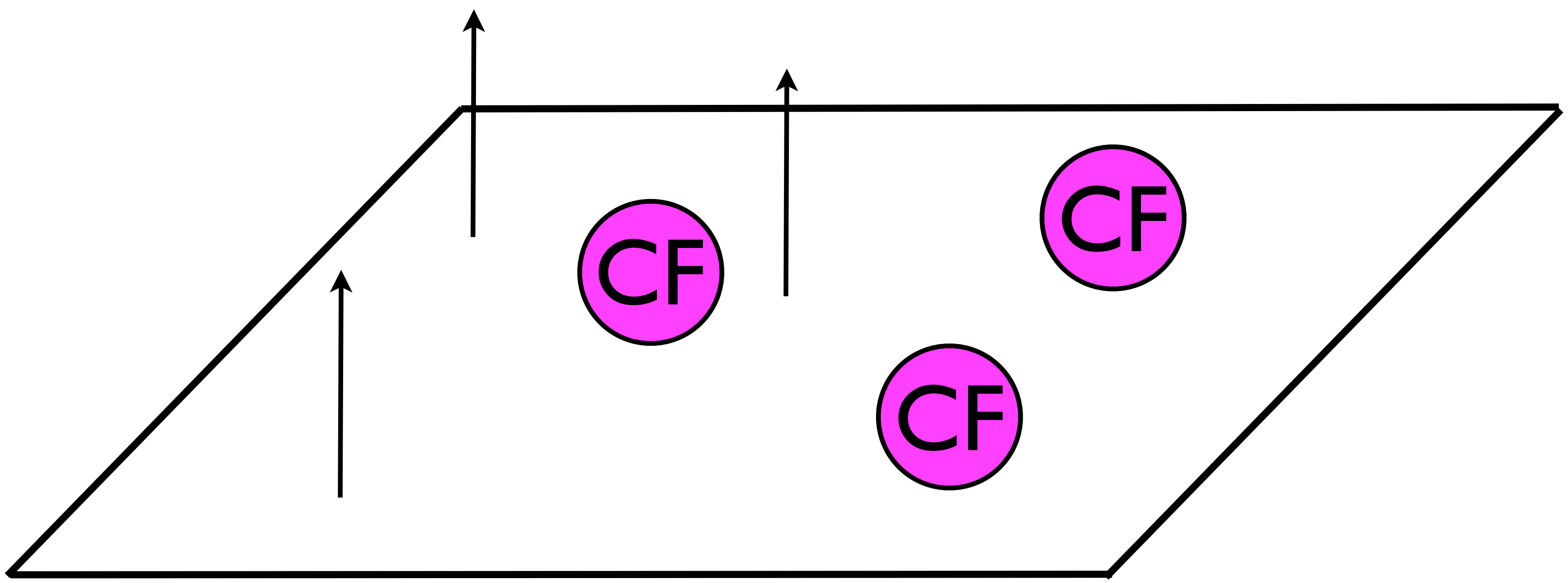

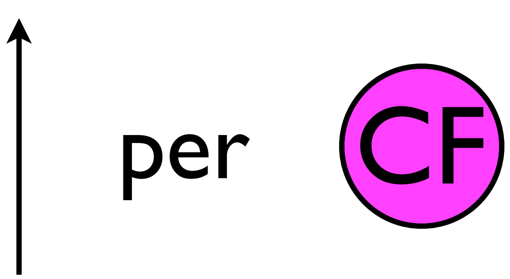

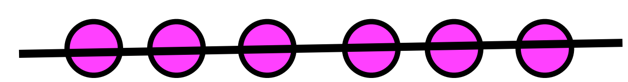

full LL: gapped state

(CF = composite fermion)

Flux attachment

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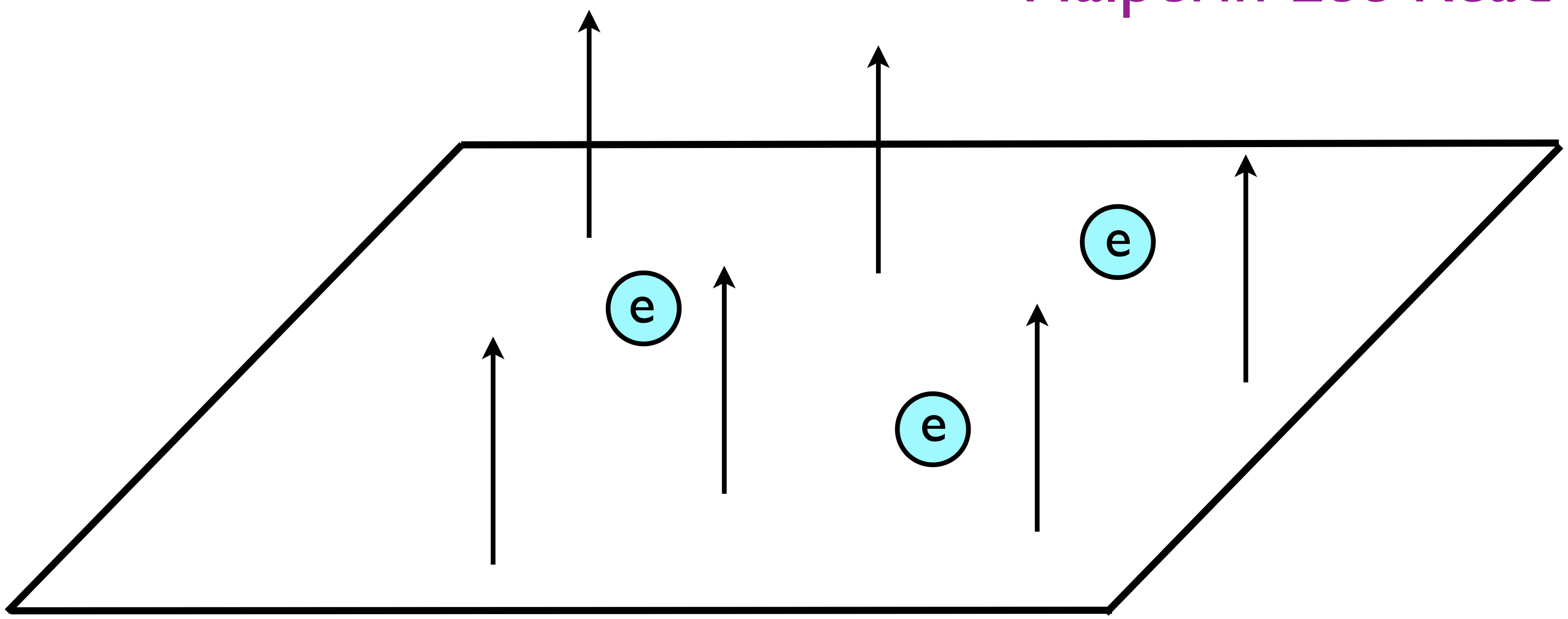
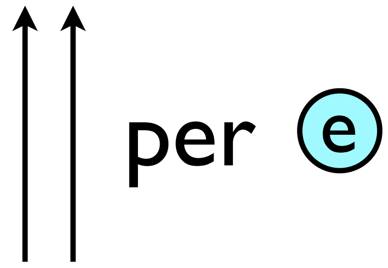


$\nu = \frac{1}{2}$



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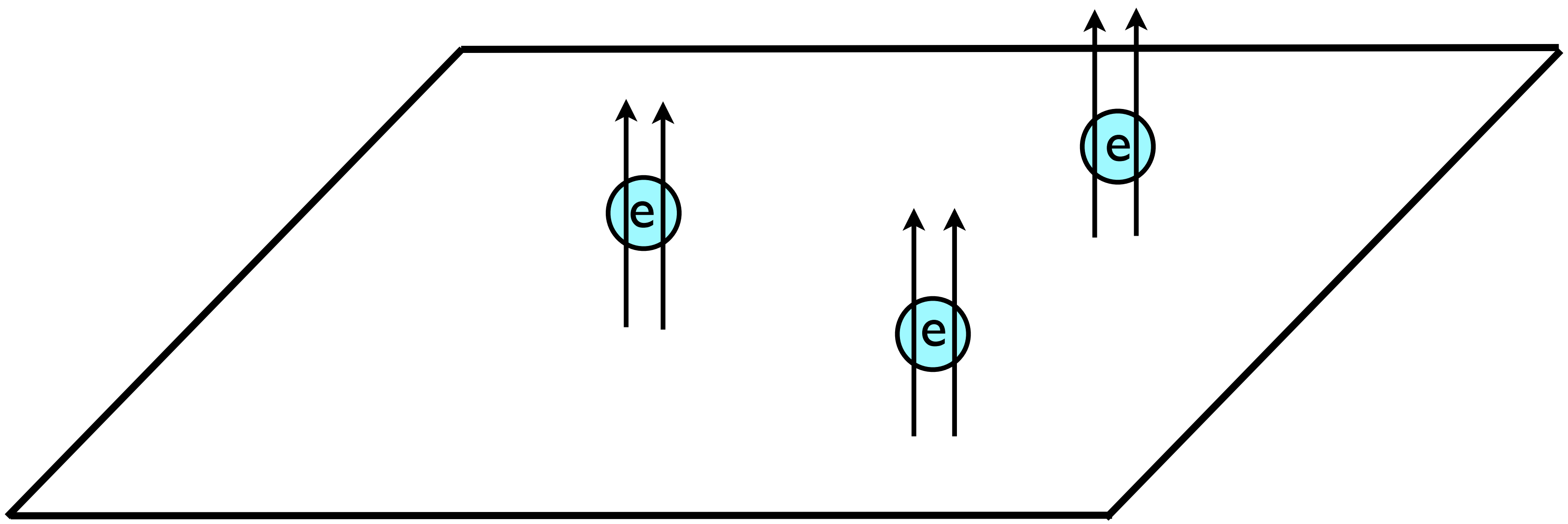
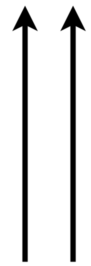


$\nu = \frac{1}{2}$  per e

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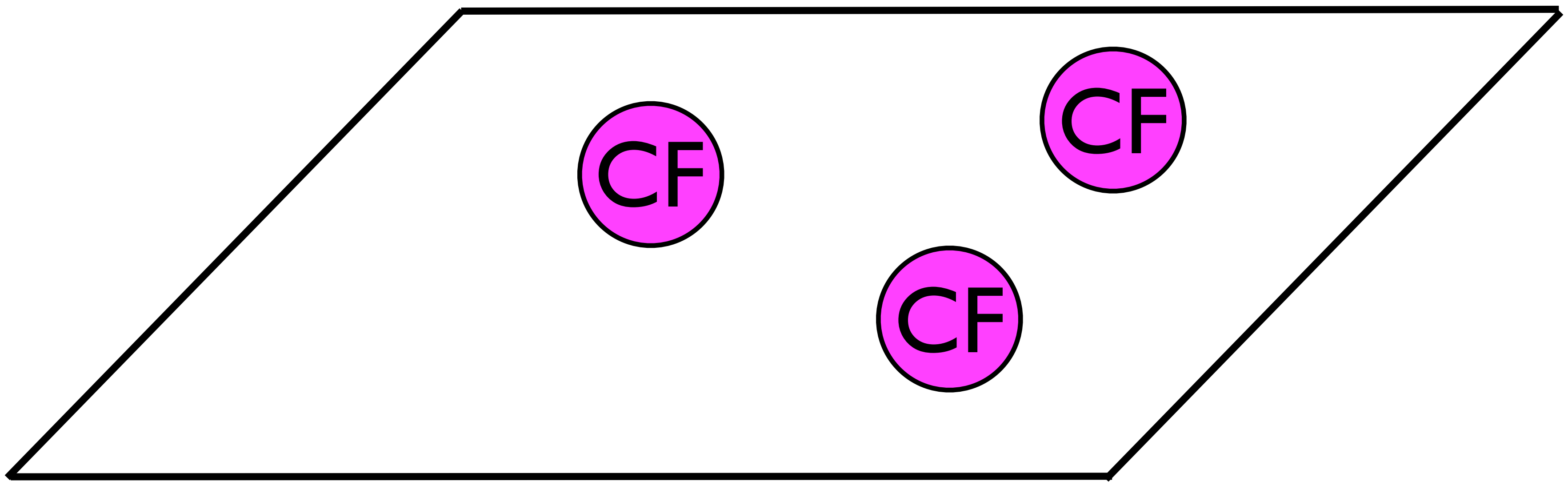

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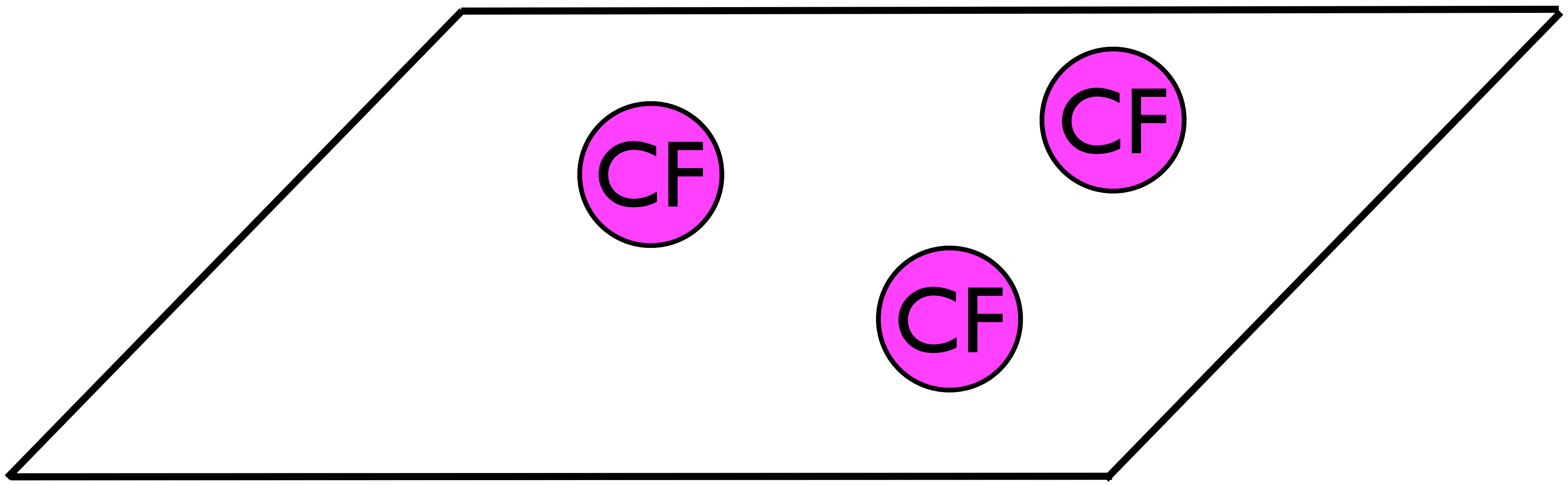


No left-over magnetic field: CFs move in straight line

- The ideas of flux attachment and composite fermion has been formalized into a field theory framework (Halperin-Lee-Read, or HLR, theory)
- very successful phenomenologically
- The composite fermion in this picture is not very different from a standard quasiparticle in condensed matter physics
  - an electron, "dressed" with two magnetic flux quanta

HLR field theory

$$\mathcal{L} = i\psi^\dagger(\partial_0 - iA_0 + ia_0)\psi - \frac{1}{2m}|(\partial_i - iA_i + ia_i)\psi|^2 + \frac{1}{2}\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda$$

$$b = \nabla \times a = 2 \times 2\pi\psi^\dagger\psi$$ “flux attachment”

mean field: $$B_{\text{eff}} = B - b = B - 4\pi n$$

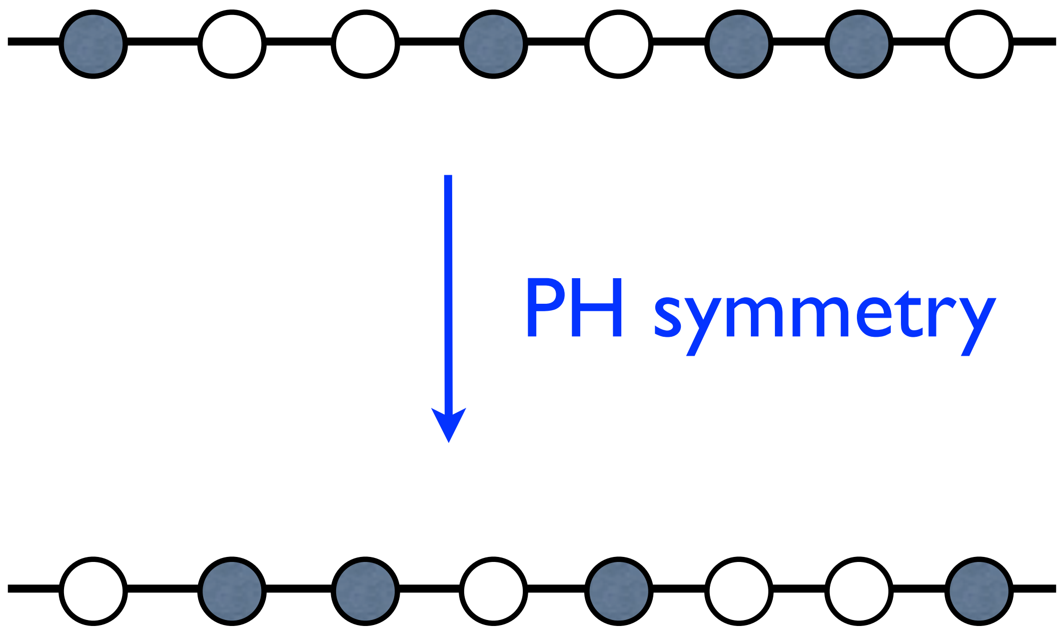
$$\nu = \frac{1}{2} \qquad\qquad B_{\text{eff}} = 0$$

low-energy dof: ψ excitations near Fermi surface

- For a long time it was thought that the HLR theory gives the correct low-energy effective theory
- There is one crucial problem

The problem of particle-hole symmetry

Particle-hole symmetry



$$\Theta|\text{empty}\rangle = |\text{full}\rangle$$

$$\Theta c_k^\dagger \Theta^{-1} = c_k$$

$$\Theta\, i\, \Theta^{-1} = -i$$

$$\nu \to 1 - \nu$$

exact symmetry the Hamiltonian on the LLL, when mixing of higher LLs negligible

$\nu = 1/2$ maps to itself

wikipedia.org

Half empty = half full

The problem

- Standard composite fermion theory attaches fluxes to particles, not holes, and does not have particle-hole symmetry
- thus it cannot be a correct low-energy description of the half-filled Landau level

Puzzle

- Particle-hole symmetry has been a puzzle in quantum Hall physics
- Composite fermion exists
- standard picture: CF = a type of "dressed electrons"
- but cannot be at the same time a "dressed hole"
- New idea from ~ 2015: use particle-vortex duality

Resolution of the problem of particle-hole symmetry

Bosonic particle-vortex duality

Peskin 1978; Dasgupta, Halperin 1981

2+1 dim

Bosonic particle-vortex duality

Peskin 1978; Dasgupta, Halperin 1981

2+1 dim

$$\mathcal{L}_1 = |\partial_\mu \phi|^2 - m^2 |\phi|^2 - \lambda |\phi|^4$$

$$\mathcal{L}_2 = |(\partial_\mu - i a_\mu)\tilde{\phi}|^2 + m^2 |\tilde{\phi}|^2 - \lambda |\tilde{\phi}|^4$$

Bosonic particle-vortex duality

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Theory 1 Theory 2

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	Theory 1	Theory 2
$m^2 < 0$	Goldstone boson	photon

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	Theory 1	Theory 2
$m^2 < 0$	Goldstone boson	photon
$m^2 > 0$	particle	vortex
	charge density	magnetic field b
	magnetic field	charge density

- Idea: electron and composite fermion are particle-vortex dual of each other

Fermionic particle-vortex duality

DTS; Metlitski, Vishwanath; Wang, Senthil 2015

Conjecture: free fermion = "QED" in 2+1 D

physical EM field

Theory 1: $$\mathcal{L} = i\bar{\psi}_e \gamma^\mu (\partial_\mu - iA_\mu)\psi_e$$

Theory 2: $$\mathcal{L} = i\bar{\psi}\gamma^\mu(\partial_\mu - ia_\mu)\psi - \frac{1}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda$$

emergent U(1) gauge field

Theory 1 ψ_{e}	Theory 2 ψ
magnetic field	density
density	magnetic field

Particle-vortex duality

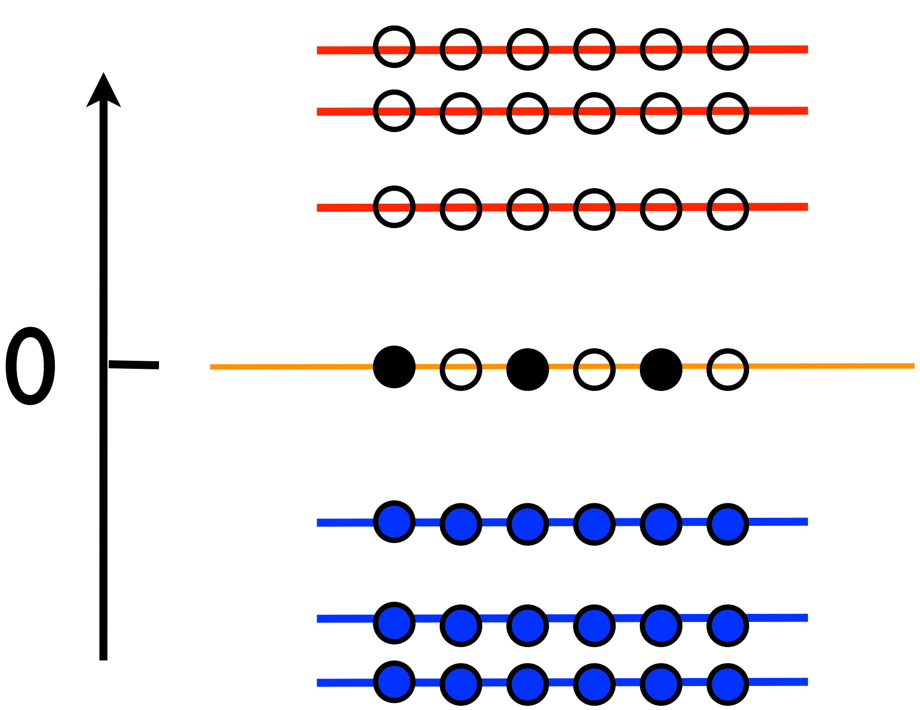
$$S = \int d^3x \, i\bar{\psi}_e \gamma^\mu (\partial_\mu - iA_\mu)\psi_e$$

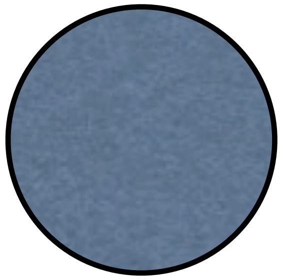
$$S = \int d^3x \left[i\bar{\psi}\gamma^\mu(\partial_\mu - ia_\mu)\psi - \frac{1}{4\pi}\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda \right]$$

$$\rho = \frac{\delta S}{\delta A_0} = -\frac{b}{4\pi} \qquad\qquad \frac{\delta S}{\delta a_0} = 0 \longrightarrow \langle \bar{\psi}\gamma^0\psi \rangle = \frac{B}{4\pi}$$

$$\|$$

$$\bar{\psi}_e\gamma^0\psi_e = \psi_e^\dagger\psi_e$$





Theory 1 in magnetic field
zero charge density

Theory 2 at finite density
and zero magnetic field

Half-filled Landau level

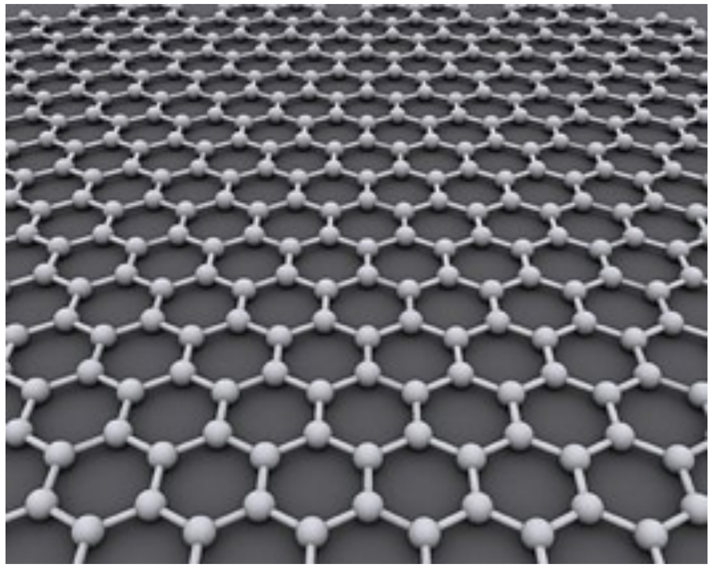
Fermi liquid

ψ_e = electrons

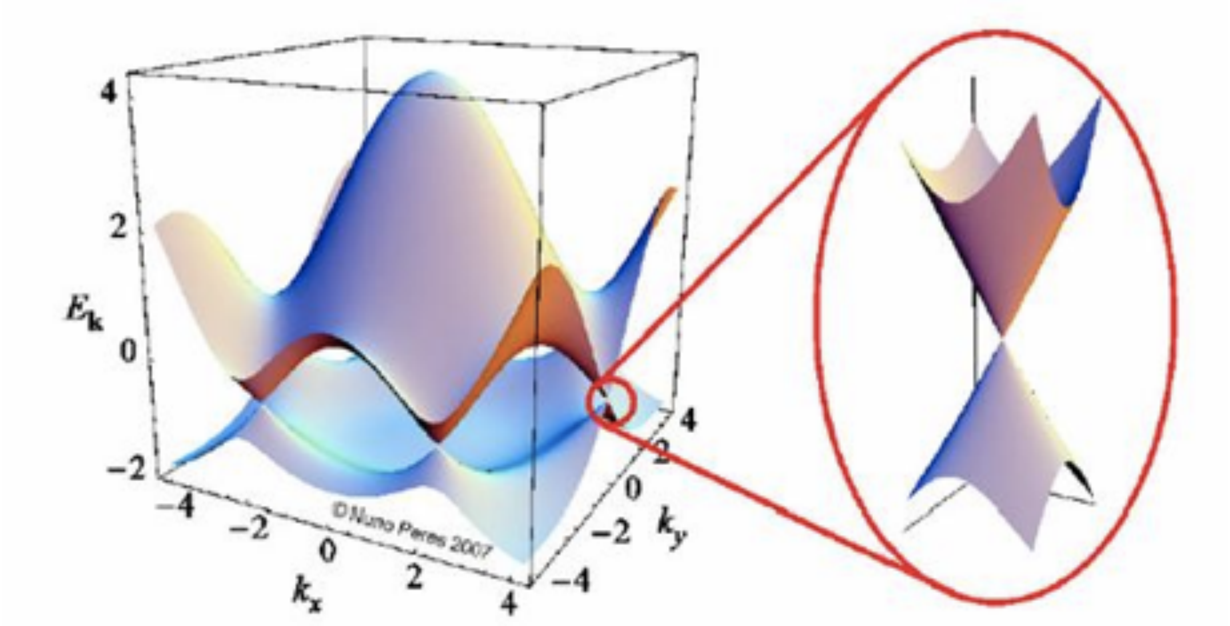
ψ = "composite fermion"

Dirac composite fermion

- Thus the quasiparticle of the half-filled Landau level is a Dirac fermion, not a nonrelativistic fermion as in the old HLR theory
- This is not the first time Dirac fermion has appeared in condensed matter physics: the most well-known case is graphene



wikipedia.



uminho

What is strange about the Dirac composite fermion

- Density of composite fermions = 1/2 the number of magnetic flux quanta
- or $n_{\mathrm{CF}} = (n_{\mathrm{e}} + n_{\mathrm{h}})/2$
- in contrast to HLR theory: $n_{\mathrm{CF}} = n_{\mathrm{e}}$
- conceptually not "dressing an electron"

- Some distinctive predictions of Dirac composite fermion theory
- The PH-Pfaffian phase

Consequences of Dirac CF

One characteristics of Fermi surface is Friedel oscillations

$$\langle O(\mathbf{x})O(\mathbf{0})\rangle \sim |\mathbf{x}|^{\alpha} e^{2ik_F x}$$

Suppression of Friedel oscillations in correlations of particle-hole symmetric observables $\hat{O} = (\rho - \rho_0)\nabla^2 \rho$

observed in numerics

Geraedts, Zaletel, Mong, Metlitsky, Vishwanath, Montrunich, 2015

Direct evidence of Berry phase π of the composite fermion

A new gapped state

- The composite fermions can form Cooper pairs
- Simplest pairing does not break particle-hole symmetry

$$\langle \epsilon^{\alpha\beta} \psi_\alpha \psi_\beta \rangle \neq 0$$

- Leads to a new state: PH-Pfaffian
- May be the observed nu=5/2 quantum Hall plateau
  Banerjee et al. Nature 2018

Observation of half-integer thermal Hall conductance

Mitali Banerjee[1], Moty Heiblum[1]*, Vladimir Umansky[1], Dima E. Feldman[2], Yuval Oreg[1] & Ady Stern[1]

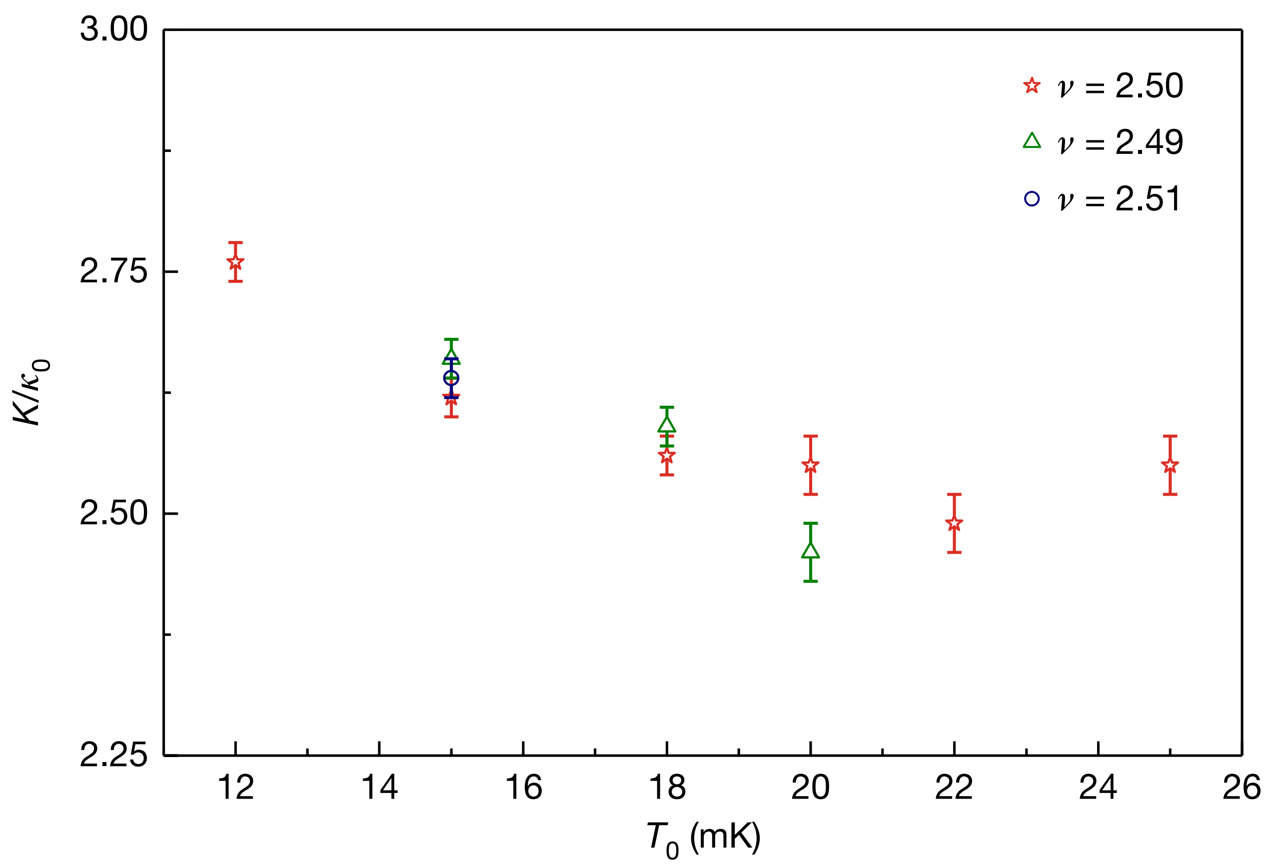


Fig. 4 | Summary of the normalized thermal conductance coefficient results for $\nu = 5/2$. Plotted is the average K/κ_0 as a function of the temperature at three different fillings on the $\nu = 5/2$ G_H conductance plateau. A clear tendency of increased thermal conductance at lower temperatures is visible. Such dependence is attributed to the increased equilibration length (among downstream and upstream modes) at lower temperatures (see ref. [26] for a similar behaviour of the $\nu = 2/3$ state). Seventeen measurements were conducted, with K/κ_0 falling in the range $K/\kappa_0 = (2.53 \pm 0.04)\kappa_0$ at electron base temperatures of $T_0 = 18$–25 mK, where most of the data points were taken.

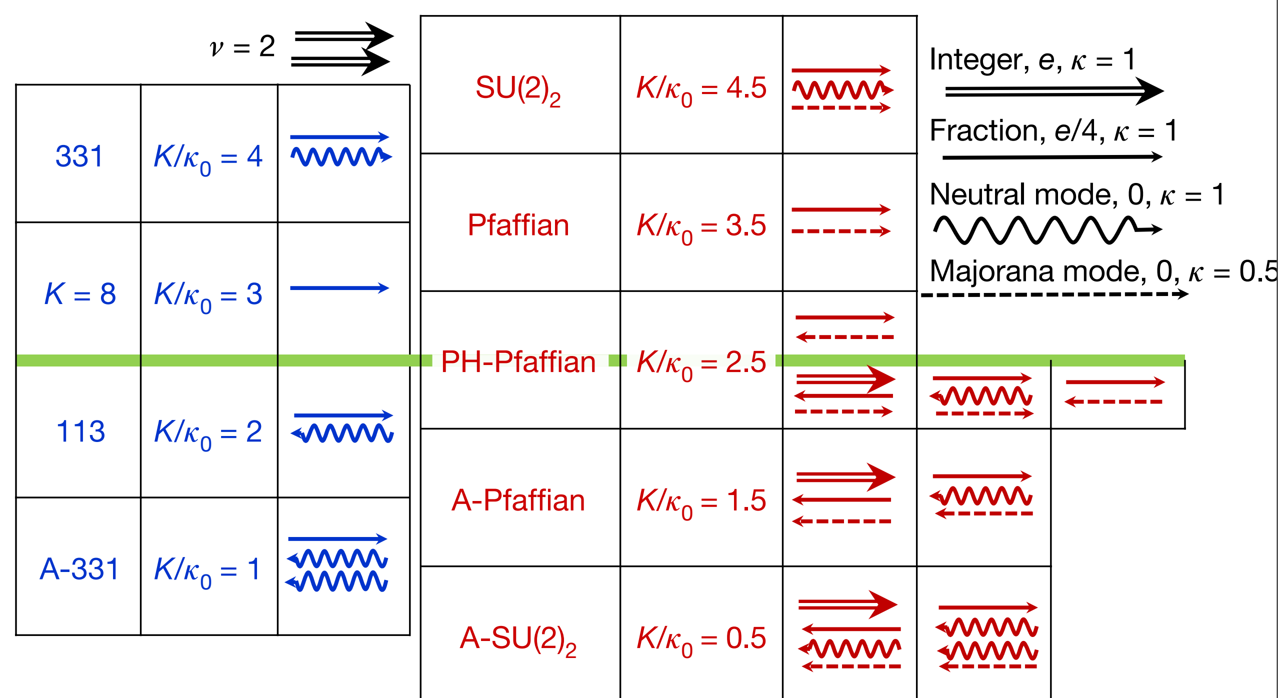
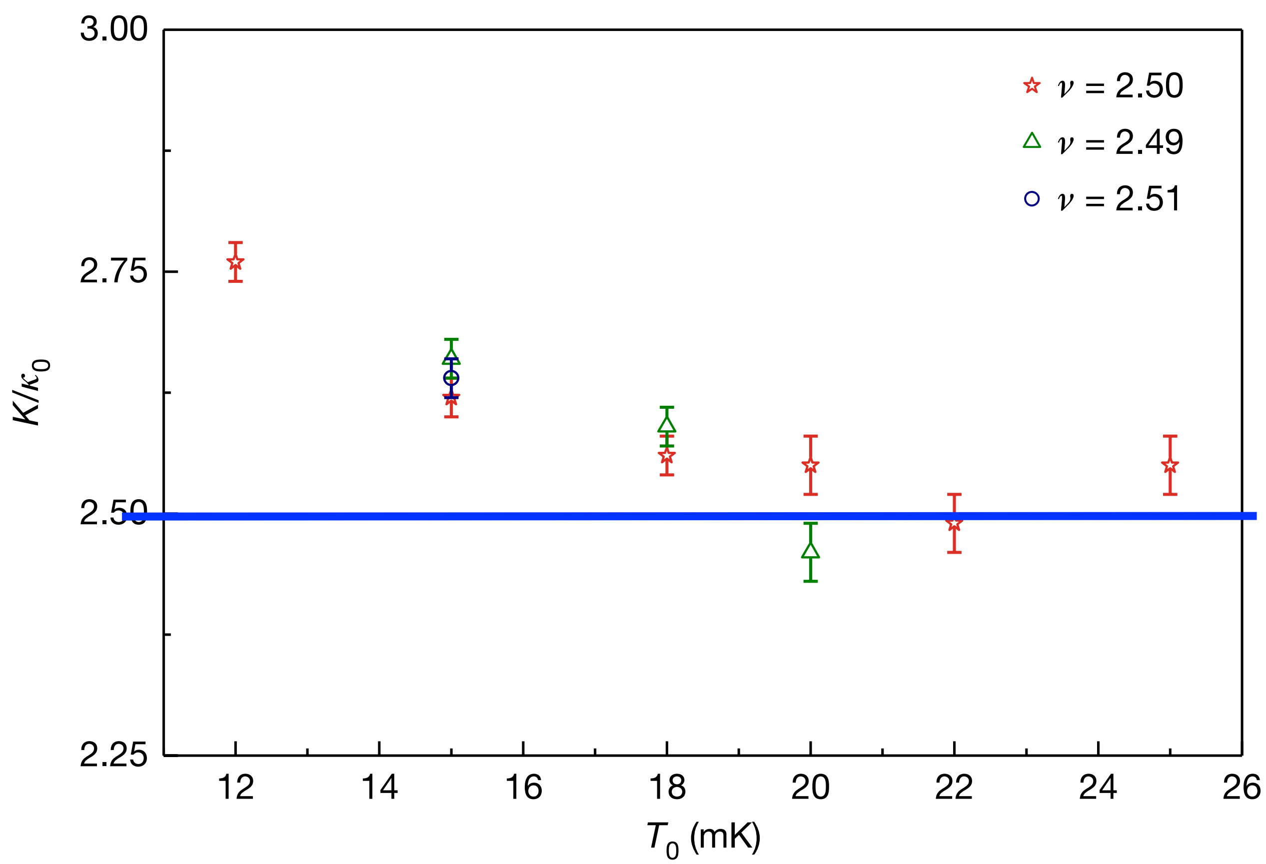


Fig. 5 | Possible orders predicted for the $\nu = 5/2$ state. Edge-mode structure of the leading candidates for the many-body state of a fractional quantum Hall $\nu = 5/2$ liquid: Pfaffian, anti-Pfaffian (A-Pfaffian) and particle–hole Pfaffian (PH-Pfaffian) topological orders and the SU(2)$_2$, $K = 8$, 331 and 113 liquids ('A' stands for 'anti'). Their expected quantized thermal Hall conductance, KT, in units of $\kappa_0 T$ are also shown. A right-pointing double-line arrow denotes a downstream edge mode of a fermion with charge $e^* = e$, contributing Hall conductivity $G_H = e^2/h$ and $K/\kappa_0 = 1$. Right- and left-pointing solid-line arrows denote a downstream and an upstream fractional charge mode, respectively, contributing $0.5G_H = e^2/(2h)$ and $K/\kappa_0 = 1$. The wavy line denotes a fermionic neutral mode with zero charge and $K/\kappa_0 = 1$, and the dashed line denotes a Majorana mode with zero charge and $K/\kappa_0 = 1/2$. A neutral mode with $K/\kappa_0 = 1$ is physically equivalent to two Majorana modes. The left (right) part of

Observation of half-integer thermal Hall conductance

Mitali Banerjee[1], Moty Heiblum[1]*, Vladimir Umansky[1], Dima E. Feldman[2], Yuval Oreg[1] & Ady Stern[1]



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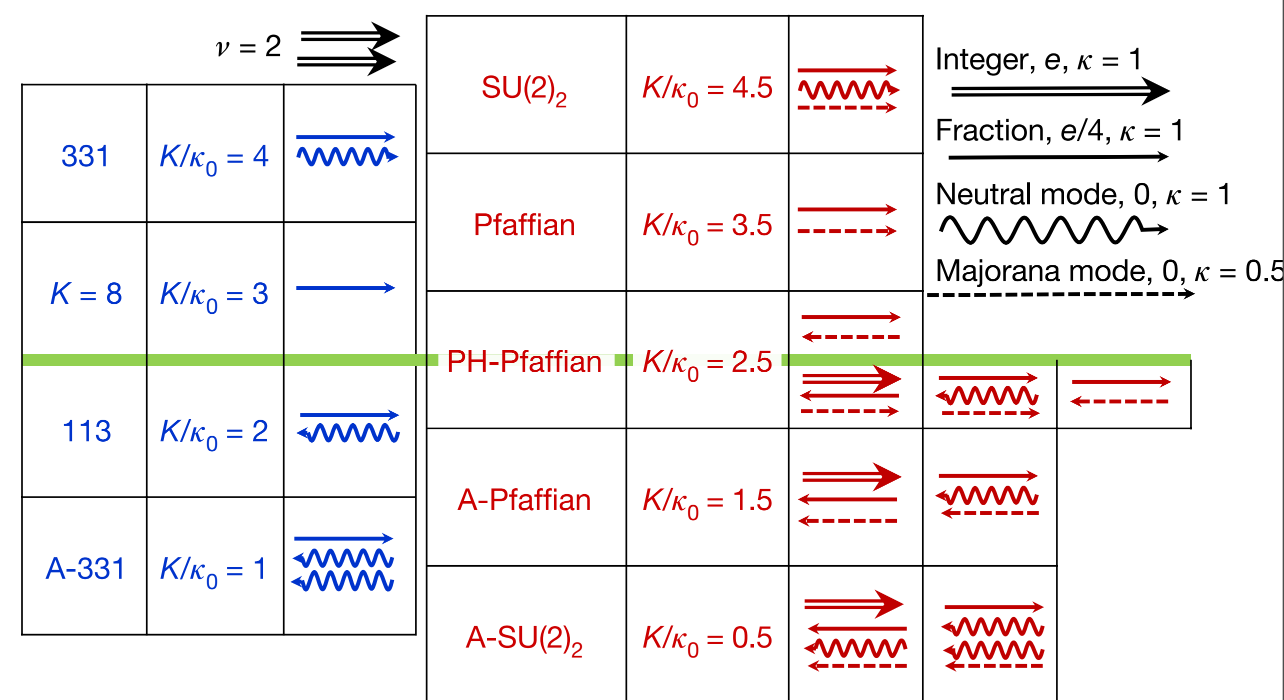

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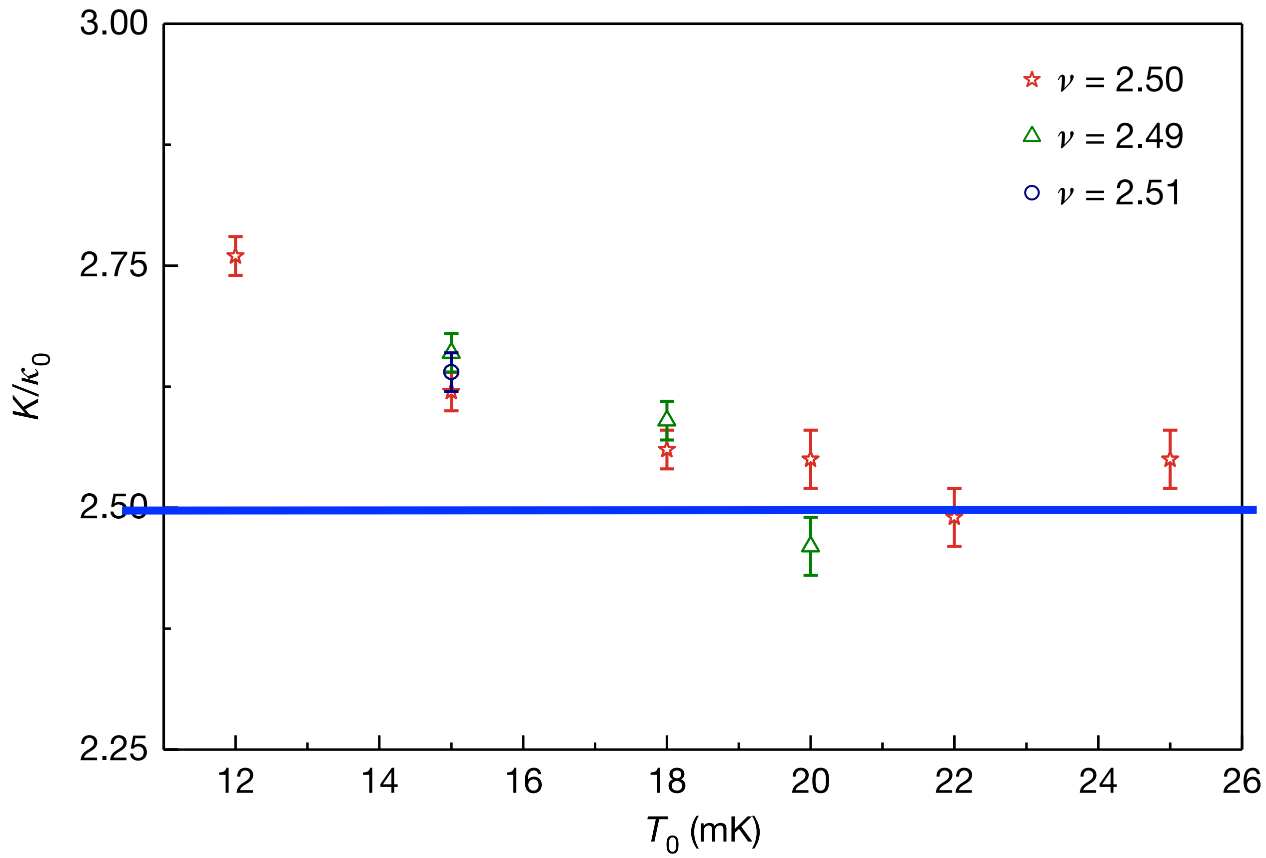

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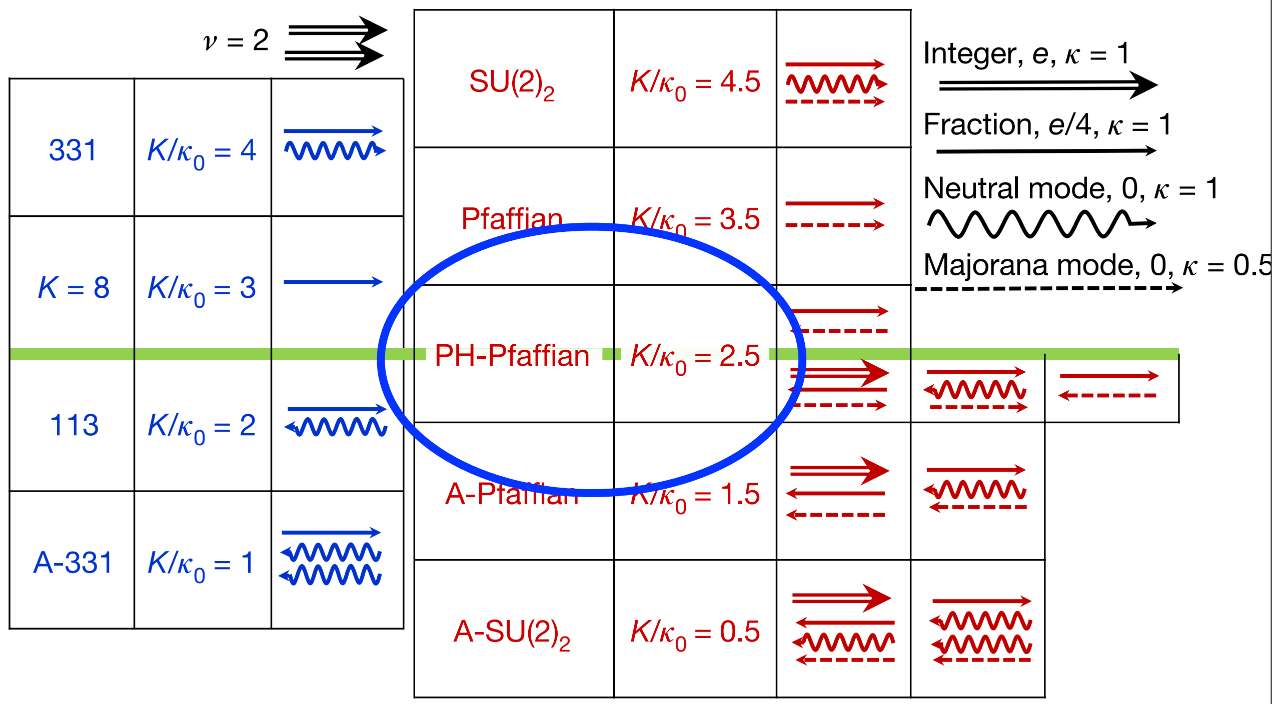

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The “seed duality”

Both the bosonic and fermionic particle-vortex duality can be derived from a “seed duality”

fermion = boson + flux

$$\mathcal{L} = L[\psi, A] - \frac{1}{2}\frac{1}{4\pi}AdA \qquad \mathcal{L} = L[\phi, a] + \frac{1}{4\pi}ada + \frac{1}{2\pi}Ada$$

From this duality, a whole “web” of new dualities can be derived

Karch, Tong; Seiberg, Senthil, Wang, Witten

Extreme small N

- It turns out that in HEP literature there has been suggestions of duality between bosonic and fermionic Chern-Simons theories
- Verified at large N but speculated to be valid also at small N Aharony 2015

Baryons, monopoles and dualities in Chern-Simons-matter theories

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Ofer Aharony

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Weizmann Institute of Science, Rehovot 7610001, Israel
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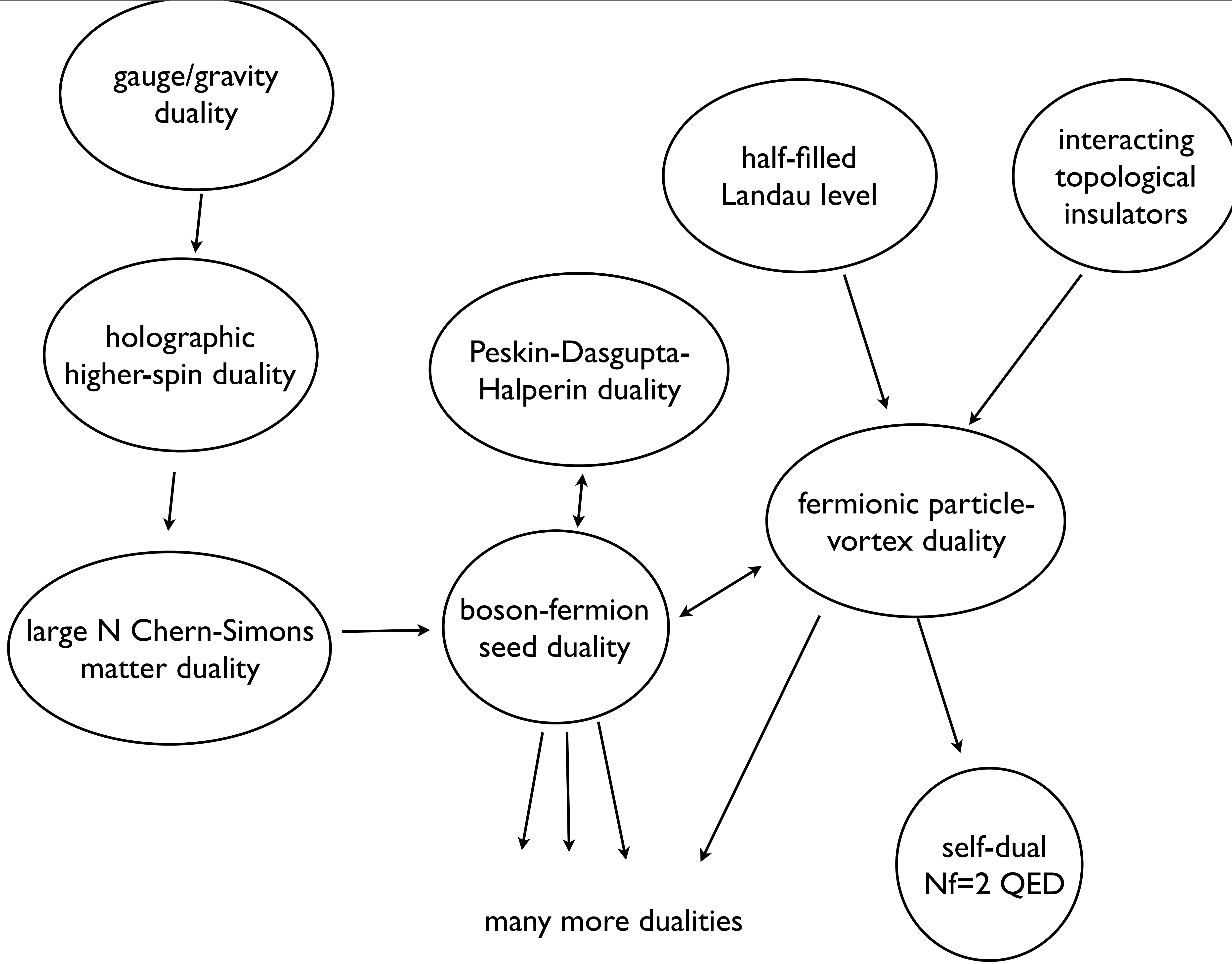
$$U(N)_{k,k} \text{ coupled to scalars} \leftrightarrow SU(k)_{-N+N_f/2} \text{ coupled to fermions,} \qquad (2.7)$$

seed duality $N = N_f = k = 1$


gauge/gravity duality
holographic higher-spin duality
large N Chern-Simons matter duality
Peskin-Dasgupta-Halperin duality
boson-fermion seed duality
half-filled Landau level
interacting topological insulators
fermionic particle-vortex duality
self-dual Nf=2 QED
many more dualities

Conclusion of part 1

- Fractional quantum Hall systems provide an experimentally realizable example of duality
- Mysterious emergence of a new type of quasiparticle; not a standard "dressed electron"
- Fruitful interaction between high-energy and condensed matter physics

End of part 1

Hydrodynamics of the composite fermions

Plan

- “Chiral metric hydrodynamics”
- Fractional quantum Hall effect
- Composite fermion
- Kelvin’s circulation theorem (1869), static structure factor
- Ref: arXiv:1907.07187
    - related work: A.Gromov & DTS “bimetric theory”, Haldane’s dynamical gravity

Hydrodynamics

- Hydrodynamics of an ideal fluid can be written in terms of the particle number density n(x) and momentum density π(x)

$$\frac{\partial n}{\partial t} + \partial_i (n v^i) = 0$$

$$\frac{\partial \pi_i}{\partial t} + \partial_j (v^j \pi_i) + \partial_i p = 0$$

Galilean invariant fluids: $\pi_i = m n v_i$

Hydrodynamics

Landau 1941

- Hydrodynamics can be formulated as a dynamical system with the Poisson brackets

$$\{\pi_i(\mathbf{x}),\, n(\mathbf{y})\} = n(\mathbf{x})\partial_i\delta(\mathbf{x}-\mathbf{y})$$
$$\{\pi_i(\mathbf{x}),\, \pi_j(\mathbf{y})\} = [\pi_j(\mathbf{x})\partial_i + \pi_i(\mathbf{y})\partial_j]\delta(\mathbf{x}-\mathbf{y})$$

- and Hamiltonian

$$H = \int d\mathbf{x}\left[\frac{1}{2m}\frac{\vec{\pi}^2(\mathbf{x})}{n(\mathbf{x})} + \epsilon(n(\mathbf{x}))\right]$$

$$\dot{n} = \{H,\, n\}$$
$$\dot{\pi}_i = \{H,\, \pi_i\}$$

Extension with tensor d.o.f.?

- We want to extend the hydrodynamics theory with a tensor d.o.f. $G_{ij}(\mathrm{x})$
- What is the natural Poisson brackets?
  - momentum density generates diffeomorphism

$$\hat{\xi} = \int d\mathbf{y}\, \xi^k(\mathbf{y})\pi_k(\mathbf{y}), \qquad x^k \to x^k + \xi^k$$

$$\{\hat{\xi},\, n(\mathbf{x})\} = -\xi^k \partial_k n - n\partial_k \xi^k$$ scalar density

$$\{\hat{\xi},\, \pi_i(\mathbf{x})\} = -\xi^k \partial_k \pi_i - \pi_k \partial_i \xi^k - \xi_i \partial_k \xi^k$$ vector density

Extending Poisson algebra

- Let's introduce a "dynamical metric" $G_{ij}(\mathrm{x})$ which transforms as a tensor

$$\{\hat{\xi},\, G_{ij}(\mathbf{x})\} = -\pounds_\xi G_{ij} = -\xi^k \partial_k G_{ij} - G_{kj}\partial_i \xi^k - G_{ik}\partial_j \xi^k.$$

- That fixes the Poisson bracket

$$\{G_{ij}(\mathbf{x}),\, \pi_k(\mathbf{y})\} = (G_{ik}(\mathbf{x})\partial_j + G_{jk}(\mathbf{x})\partial_i + \partial_k G_{ij})\delta(\mathbf{x}-\mathbf{y})$$

- Next: {G, G}
- {G, G} = 0: theory of a solid

Chiral metric hydro

Alternative in 2 spatial dimensions

$$\{G_{ij}(\mathbf{x}),\, G_{kl}(\mathbf{y})\} = -\frac{1}{s}(\varepsilon_{ik}G_{jk} + \varepsilon_{il}G_{jk} + \varepsilon_{jk}G_{il} + \varepsilon_{jl}G_{ik})\delta(\mathbf{x}-\mathbf{y})$$

- We can consistently impose (det G)$^{1/2}$ = n
- Hydrodynamics equations

$$\partial_t n = -\partial_i(nv^i),$$

$$\partial_t \pi_i = -n\partial_i\mu - \pi_j\partial_i v^j - \partial_j(v^j\pi_i) - \partial_j(\sigma^{jk}G_{ki}) + \frac{1}{2}\sigma^{jk}\partial_i G_{jk}\,,$$

$$\partial_t G_{ij} = -v^k\partial_k G_{ij} - G_{ik}\partial_j v^k - G_{jk}\partial_i v^k + \frac{1}{s}(\epsilon_{ik}G_{jl} + \epsilon_{jk}G_{il})\sigma^{kl}.$$

$$\delta H = \int d\mathbf{x}\, \left[\mu(\mathbf{x})\delta n(\mathbf{x}) + v^i(\mathbf{x})\delta\pi_i(\mathbf{x}) + \frac{1}{2}\sigma^{ij}(\mathbf{x})\delta G_{ij}(\mathbf{x})\right],$$

Normal modes

- Linearizing the equations one finds usual sound wave and shear mode, and a gapped spin-2 mode

$$G_{ij} = n(\delta_{ij} + Q_{ij})$$

$$Q_{xx} = -Q_{yy} \sim \cos \omega t$$

$$Q_{xy} \sim \sin \omega t$$

$$\omega = \frac{2\mu}{ns}$$

"Lamé constant"

At small frequencies, a fluid with Hall viscosity

$$\eta^H = \frac{sn}{2}$$

s = average "orbital spin"

Relevance to quantum Hall effect?

Particle-vortex duality

original fermion	composite fermion
magnetic field	density
density	magnetic field

$$\mathcal{L} = i\bar{\psi}\gamma^\mu(\partial_\mu - ia_\mu)\psi - \frac{1}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda + \frac{1}{8\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda$$

$$\rho = \frac{\delta S}{\delta A_0} = \frac{B-b}{4\pi}$$

$$\frac{\delta S}{\delta a_0} = 0 \longrightarrow \langle\bar{\psi}\gamma^0\psi\rangle = \frac{B}{4\pi}$$

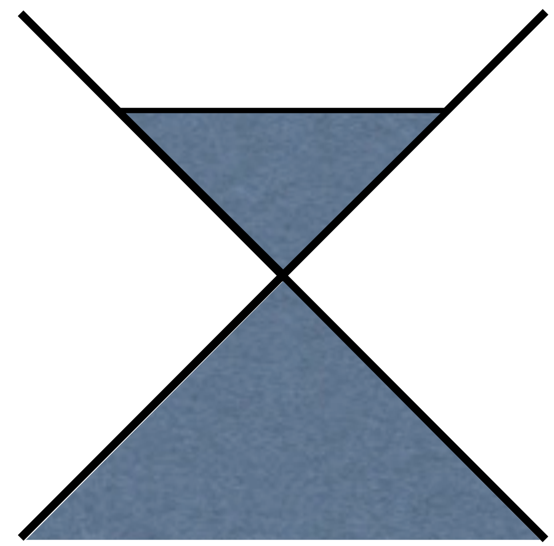
Mapping from electrons to CFs

electron

$$\nu = \frac{1}{2}, \quad B \neq 0$$

half-filled Landau level

composite fermion

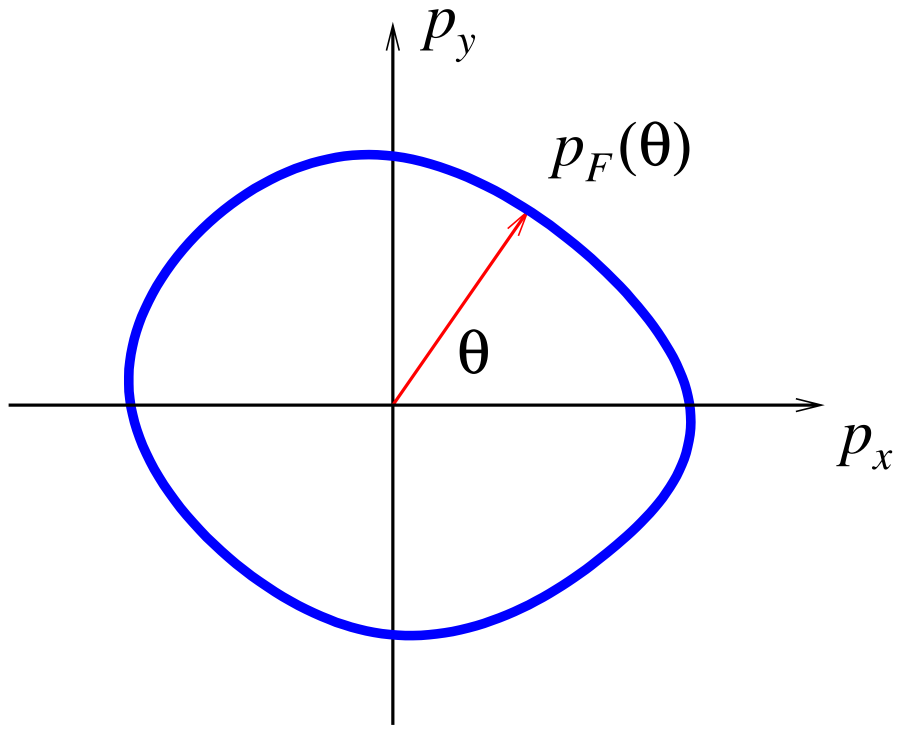


$$\rho \neq 0 \qquad b = 0$$

Fermi liquid of CFs

Deviation from half filling → CF in b field

Bosonic excitations



Low-energy, long-wavelength excitations: fluctuations of the shape of the Fermi surface

$$p_F(t, \mathbf{x}, \theta) = p_F^0 + \sum_{n=-\infty}^{\infty} u_n(t, \mathbf{x})\, e^{-in\theta}.$$

One scalar field per spin

At low momenta we can limit ourselves to a few lowest modes

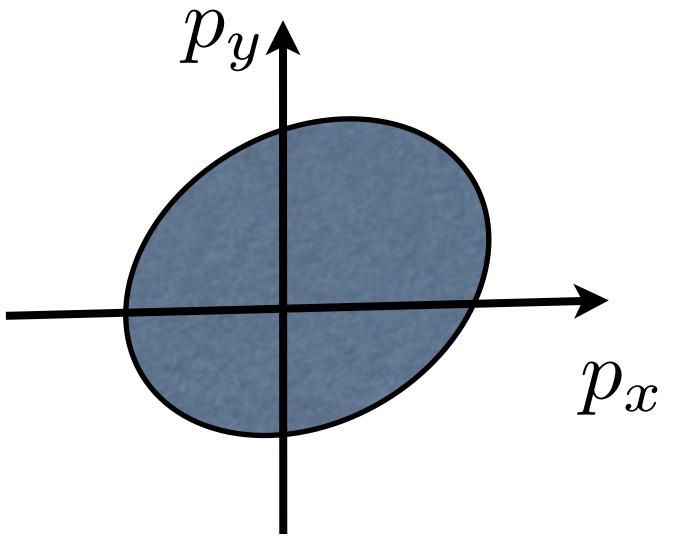
$$q\ell_B \ll 1/N \qquad \nu = \frac{N}{2N+1}$$

(length >> CF semiclassical orbit)

Nematic hydrodynamics

- Degrees of freedom:
  - density

$$n(\mathbf{x}) = \int \frac{d\mathbf{p}}{(2\pi)^2} f(\mathbf{x}, \mathbf{p})$$



  - momentum density $\pi_i(\mathbf{x}) = \int \frac{d\mathbf{p}}{(2\pi)^2} p_i f(\mathbf{x}, \mathbf{p})$

  - effective metric

$$\int \frac{d\mathbf{p}}{(2\pi)^2} p_i p_j f(\mathbf{x}, \mathbf{p}) = \frac{\pi_i \pi_j}{n} + \pi n(\mathbf{x}) G_{ij}(\mathbf{x})$$

$$\sqrt{\det G} = n$$

Poisson brackets

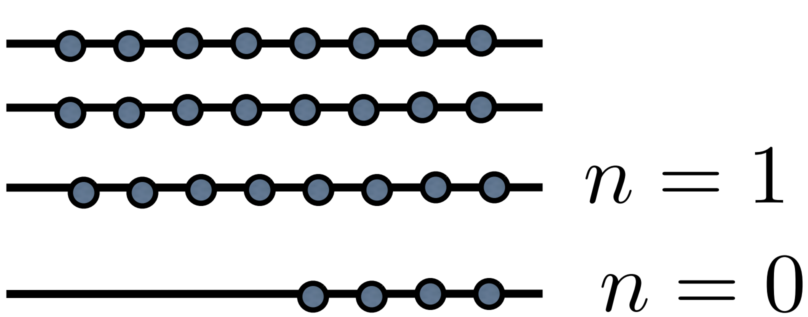
$$\{p_i,\, p_j\} = -\epsilon_{ij} b$$

$$\{G_{ij}(\mathbf{x}),\, G_{kl}(\mathbf{y})\} = -\frac{1}{s}(\varepsilon_{ik}G_{jk} + \varepsilon_{il}G_{jk} + \varepsilon_{jk}G_{il} + \varepsilon_{jl}G_{ik})\delta(\mathbf{x}-\mathbf{y})$$

s can be determined from the Hall viscosity

= average "orbital spin" of composite fermion

$$s = \frac{1}{N+\frac{1}{2}}\left(\frac{1}{2}\cdot 0 + 1 + 2 + \cdots + N\right) = \frac{N(N+1)}{2N+1}\,.$$



Hydrodynamic equation

CF dipole moment

$$H = H_0[n, \pi_i, G_{ij}] + \int d\mathbf{x} \left(-a_0 n + \frac{\varepsilon^{ij} E_j}{B} \pi_i \right)$$

$$\dot{A} = \{H,\, A\}$$

$$\delta H_0 = \int d\mathbf{x} \left(\mu \delta n + u^i \delta \pi_i + \tfrac{1}{2} \tau^{ij} \delta G_{ij} \right)$$

$$\dot{n} + \partial_i (n v^i) = 0$$

$$v^i = u^i + \frac{\varepsilon^{ij} E}{B}$$

$$\dot{\pi}_i = n e_i + \epsilon_{ij} n v^j b + \pi_j \partial_j v^j - \partial_j (v^j \pi_i) - n \partial_i \mu$$

$$- \partial_j (\tau^{jk} G_{ki}) + \frac{1}{2} \tau^{jk} \partial_i G_{jk}$$

$$\dot{G}_{ij} = \cdots$$

Prescription for response

- Find a solution to the hydrodynamic equations which satisfy the constraints:

$$n = \frac{B}{4\pi}, \qquad nv^i = \frac{1}{4\pi}\varepsilon^{ij}E_j$$

- Read out electron density and current

$$\rho = \frac{B-b}{4\pi} - \epsilon^{ij}\partial_i\left(\frac{\pi_j}{B}\right)$$

$$= n - \frac{b+\omega}{4\pi}$$

$$\omega = \vec{\nabla} \times \left(\frac{\vec{\pi}}{n}\right)$$

vorticity

GMP algebra

$$\rho = \frac{B-b}{4\pi} - \epsilon^{ij}\partial_i\left(\frac{\pi_j}{B}\right)$$

$$\{\pi_i(\mathbf{x}),\, \pi_j(\mathbf{y})\} = [\pi_j(\mathbf{x})\partial_i + \pi_i(\mathbf{y})\partial_j - \varepsilon_{ij}bn]\delta(\mathbf{x}-\mathbf{y})$$

$$[\rho(\mathbf{p}),\, \rho(\mathbf{q})] = \ell_B^2(\mathbf{p}\times\mathbf{q})\rho(\mathbf{p}+\mathbf{q})$$

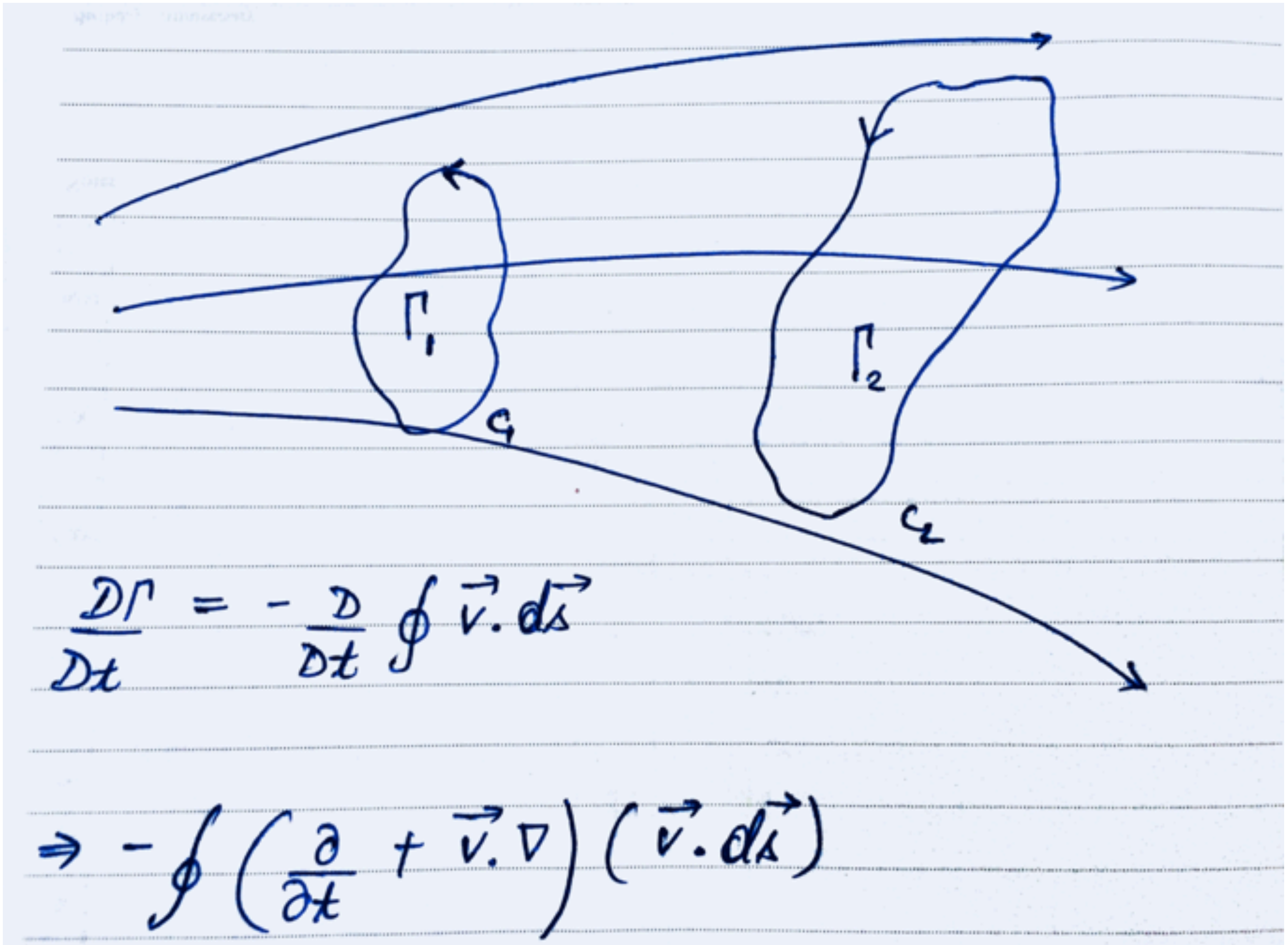
long-distance version of the GMP algebra

Girvin, MacDonald, Platzman 1986

CF theory “knows” about LLL projection

Kelvin’s circulation theorem

1869



engineering.stackexchange.com

- In ideal hydrodynamics there is an infinite number of conserved quantities

vorticity

$$I_F = \int d\mathbf{x}\, n(\mathbf{x}) F\left(\frac{\omega(\mathbf{x})}{n(\mathbf{x})}\right), \qquad \omega = \vec{\nabla} \times \left(\frac{\vec{\pi}}{n}\right)$$

- Property of Poisson algebra, not of Hamiltonian: I_F commutes with all hydrodynamic variables

$$\dot{\omega} + \vec{\nabla} \cdot (\omega \vec{v}) = 0$$

Kelvin's circulation theorem

- In the presence of magnetic field and metric degree of freedom, what is conserved is

$$\Omega = b + \omega + \frac{s}{2}\sqrt{G}R[G] \qquad \dot{\Omega} + \vec{\nabla} \cdot (\Omega \vec{v}) = 0$$

- zero PB with other hydro variables

Electron density is curvature

- Electron density

$$\rho_{\mathrm{e}} = \frac{\delta S}{\delta A_0} = \frac{B-b}{4\pi} - \epsilon^{ij}\partial_i\left(\frac{\pi_j}{B}\right).$$

dipole $\vec{\nabla}\cdot\vec{d}$

$$= \frac{B-\Omega}{4\pi} + \frac{s}{8\pi}\sqrt{G}\,R[G].$$

$$\delta\rho_{\mathrm{e}} = \frac{s}{8\pi}\sqrt{G}\,R[G].$$

An immediate consequence

$$\delta\rho_e = \frac{s}{8\pi}\sqrt{G}R[G] \quad \sim \partial_i \partial_j G_{ij}$$

$$\to \langle \delta\rho_e \delta\rho_e \rangle_{\omega,q} \sim q^4$$

Property of the lowest Landau level

- Since R ~ ∂∂G: density-density correlation functions ~ q^4
- The static structure factor is independent of H

$$G_{ij} = \delta_{ij} + h_{ij}$$

$$\boxed{\langle \delta\rho_{\mathrm{e}} \delta\rho_{\mathrm{e}} \rangle_q = \frac{N(N+1)}{2N+1} \frac{q^4}{16\pi B}\,.}$$

- Since R ~ ∂∂G: density-density correlation functions ~ q^4
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$$G_{ij} = \delta_{ij} + h_{ij} \qquad [h_{\bar{z}\bar{z}}(\mathbf{x},\, h_{zz}(\mathbf{y})] = \frac{1}{s}\delta(\mathbf{x}-\mathbf{y})$$

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a $a^\dagger$

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$$a \qquad a^{\dagger}$$

$$\delta\rho_{\rm el} \sim \partial_z^2 h_{\bar{z}\bar{z}} + \partial_{\bar{z}}^2 h_{zz}$$

$$\boxed{\langle \delta\rho_{\rm e}\delta\rho_{\rm e}\rangle_q = \frac{N(N+1)}{2N+1}\frac{q^4}{16\pi B}\,.}$$

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should be good for large *N*
for *N*=1 (nu=1/3, 2/3): reproduces exact value from Laughlin wf

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should be good for large *N*
for *N*=1 (nu=1/3, 2/3): reproduces exact value from Laughlin wf

(HLR: 1/2 or 2 times Laughlin's value)

Low-q magnetoroton

- The magnetoroton at small momenta is then the excitation of the "dynamical metric" G_{ij}
- spin 2 directed opposite to magnetic field for $\nu=N/(2N+1)$, along magnetic field for $\nu=(N+1)/(2N+1)$
- spin in principle detectable by polarized Raman scattering
  - (distinguishes Pfaffian and anti-Pfaffian)

- The calculation can be extended to nu=1/4 where there is no symmetry arguments fixing the Berry phase Fradkin, Goldman; Chong Wang, Senthil; Jie Wang
- allow one to read out the Berry phase from the static structure factor on Jain states near 1/4 (Dung X. Nguyen, DTS to appear)

Static structure factor

- Equal time density-density correlation function

$$s(q) = \frac{1}{\rho_0}\int d\mathbf{x}\, e^{-i\mathbf{q}\cdot\mathbf{x}}\langle\rho(t,\mathbf{x})\rho(t,\mathbf{0})\rangle$$

if restricted to LLL states: "projected static structure factor"

$$\bar{s}(q) = s(q) - (1 - e^{-q^2/2})$$

For gapped states at small q

$$\bar{s}(q) = \bar{s}_4(q\ell_B)^4 + O(q^6)$$

Can be read out from the wave function

Berry phase

$$\lim_{N\to\infty}\left[\bar{s}_4\left(\frac{N}{4N+1}\right)-\bar{s}_4\left(\frac{N}{4N-1}\right)\right]=\frac{\varphi}{4\pi}$$

Putting *N*=1 and using Laughlin wavefuncitons $\varphi=\pi$

Conclusion

- Low-q regime of FQH liquid: described by a fluid with internal metric degree of freedom, coupled to a gauge field
- Electron density ~ curvature of dynamic metric
- Static structure factor: algebraic calculation

End of part 2